Photometric Solutions via one-step multilinear least squares fit

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1 Introduction

For sake of simplicity, let's consider a case where one single passband is used. If f is the flux integrated over the PSF and t_{exp} is the exposure time, then the instrumental magnitude m in that passband is defined as follows:

$$m = -2.5\ log(f) + 2.5\ log(t_{exp}) \tag{1}$$

The conversion to the magnitude M of some photometric system has to take into account two items:

- correction for the atmospheric extinction;
- correction for the difference between the photometric system passband and the real passband;

This is usually achieved using the following law (see, for example, Harris et al. 1981):

$$M = m + zp + (\gamma + \gamma' \times c) \times c - (k + k' \times c) \times z \tag{2}$$

where zp is the zero point, γ and γ' are the colour-terms, c is one of the colours of the object (in the same photometric system), k and k' are the extinction coefficients in that passband and z is the airmass. For example, in the case of B Johnson, one could have:

$$B = b + zp_B + \gamma_B \times (B - V) + \gamma'_B \times (B - V)^2 - \{k_B + k'_B \times (B - V)\} \times z \tag{3}$$

For the full solutions and many other aspects of the problem, the reader should refer to the extensive work by Harris et al. (1981). Note that their general algorithm is implemented in the *IRAF* command `fitparams` (package `noao.digiphot.photcal`).

For sake of simplicity, hereafter we will drop the second order colour term and the first order airmass term ($\gamma' = k' = 0$), i.e. we will consider the following model for our photometric observations:

$$M = m + zp + \gamma \times c - k \times z \tag{4}$$

Let's now focus on the opposite problem, i.e. the determination of the zero point. This is usually done observing several photometric standard stars, for which all required quantities are known. Due to the measurement errors, each standard star would give a different zero point:

$$zp_i = M_i - m_i - \gamma \times c_i + k \times z_i \tag{5}$$

If γ and k are known, then one can compute all zp_i $(i = 1, 2, ..., N)$ and take their weighted mean as the best estimate for the zero point:

$$zp = \frac{\sum_{i=1}^{N} zp_i \; w_i}{\sum_{i=1}^{N} w_i} \tag{6}$$

If we assume that the errors on zp_i are dominated by the errors σ_i on the measured intrumental magnitudes m_i, the weights are simply given by $w_i = 1/\sigma_i^2$.

Of course, if one does not have k and γ, then he has to compute them. This is often achieved by a two–step process. First, observations of the same standard stars at two (or more) different airmasses are used to compute the extinction coefficients, which are then adopted to correct the observations and to derive the colour terms from stars with different colours.

In the next section we describe a simple way of doing it in in a single step, when *different* standard stars (with different colours) are observed at different airmasses. The solution is a one–step multilinear least squares fit to the observed data.

2 Computing k and γ

Before starting, let's set $d_i = M_i - m_i$, so that d_i is the difference between the real magnitude and the instrumental magnitude of the $i - th$ photometric standard star. Given what we have mentioned in the previous section, we expect that the observed differences obey to the following law:

$$d = zp + \gamma \times c - k \times z \tag{7}$$

Then, γ and k are obtained through a multilinear least squares fit of this relation to the observed differences. As usual, we define χ^2 as follows:

$$\chi^2 = \sum_{i=1}^{N} \left(\frac{d_i - (zp + \gamma \times c_i - k \times z_i)}{\sigma_i} \right)^2 \tag{8}$$

where σ_i are the RMS errors on the values of m_i, always under the assumption that the errors on d_i are dominated by the errors on m_i. The minimum of χ^2 is reached when the following conditions are fullfilled:

$$\begin{cases} \frac{\partial \chi^2}{\partial zp} = 0 \\ \\ \frac{\partial \chi^2}{\partial k} = 0 \\ \\ \frac{\partial \chi^2}{\partial \gamma} = 0 \end{cases} \tag{9}$$

where $W = \sum w_i \equiv \sum 1/\sigma_i^2$. The first equation of Eqs. 9 can be esplicitely rewritten as:

$$zp = \frac{1}{W} \sum_{i=1}^{N} (d_i - \gamma \times c_i + k \times z_i) \; w_i \tag{10}$$

which, as expected, gives Eq. 6. Now, we can also write:

$$zp = \sum_{i=1}^{N} \frac{d_i w_i}{W} - \gamma \sum_{i=1}^{N} \frac{c_i w_i}{W} + k \sum_{i=1}^{N} \frac{z_i w_i}{W} \tag{11}$$

or, in a more compact format:

$$zp = \bar{d} - \gamma \bar{c} + k \bar{z} \tag{12}$$

with obvious meaning for barred quatities. If now we set $\Delta c_i = c_i - \bar{c}$, $\Delta z_i = z_i - \bar{z}$ and $\Delta d_i = d_i - \bar{d}$ we can rewrite Eq. 8 as follows:

$$\chi^2 = \sum_{i=1}^{N} (\Delta d_i - \gamma \Delta c_i + k \Delta z_i)^2 \; w_i \tag{13}$$

then, computing the partial derivatives with respect to k and γ gives:

$$\begin{cases} \frac{\partial \chi^2}{\partial \gamma} = 2 \sum_{i=1}^{N} (\Delta d_i - \gamma \Delta c_i + k \Delta z_i) \; w_i \; \Delta c_i = 0 \\ \\ \frac{\partial \chi^2}{\partial k} = 2 \sum_{i=1}^{N} (\Delta d_i - \gamma \Delta c_i + k \Delta z_i) \; w_i \; \Delta z_i = 0 \end{cases} \tag{14}$$

To simplify further calculations let's make the following settings:

$$\begin{array}{rcl} S_{dc} & = & \sum \Delta d_i \Delta c_i \; w_i \\ S_{dz} & = & \sum \Delta d_i \Delta z_i \; w_i \\ S_{cc} & = & \sum \Delta c_i^2 \; w_i \\ S_{cz} & = & \sum \Delta c_i \Delta z_i \; w_i \\ S_{zz} & = & \sum \Delta z_i^2 \; w_i \end{array} \tag{15}$$

then substituting in Eqs. 14 we have:

$$\begin{cases} \gamma S_{cc} - k S_{cz} = S_{dc} \\ \\ \gamma S_{cz} - k S_{zz} = S_{dz} \end{cases} \tag{16}$$

Then, if we set

$$\Delta = S_{cc} S_{zz} - S_{cz}^2 \tag{17}$$

we finally get the solutions we have been looking for:

$$\gamma = \frac{S_{dc} S_{zz} - S_{cz} S_{dz}}{\Delta} \tag{18}$$

$$k = \frac{S_{dc} S_{cz} - S_{cc} S_{dz}}{\Delta} \tag{19}$$

These two equations, together with Eq. 12, solve completely the problem.

2.1 Uncertainties on the photometric solution

We want now to estimate the uncertainties on the photometric solution. If σ_i are the RMS errors on the measurements, then they propagate into k and γ as usual:

$$\left\{ \begin{array}{l} \sigma_\gamma^2 = \sum_{i=1}^N \sigma_i^2 \, \left(\frac{\partial \gamma}{\partial d_i}\right)^2 \\ \\ \sigma_k^2 = \sum_{i=1}^N \sigma_i^2 \, \left(\frac{\partial k}{\partial d_i}\right)^2 \end{array} \right. \tag{20}$$

Taking into account the fact that we have:

$$\left\{ \begin{array}{l} \frac{\partial \gamma}{\partial d_i} = \frac{1}{\Delta}\left(S_{zz} c_i w_i - S_{cz} z_i w_i\right) \\ \\ \frac{\partial k}{\partial d_i} = \frac{1}{\Delta}\left(S_{cz} c_i w_i - S_{cc} z_i w_i\right) \end{array} \right. \tag{21}$$

the solution of Eqs. 20 gives the following expressions for the RMS uncertainties on γ and k:

$$\sigma_\gamma^2 = \frac{S_{zz}}{\Delta} \quad ; \quad \sigma_k^2 = \frac{S_{cc}}{\Delta} \tag{22}$$

from which it is clear that the solution becomes more and more uncertain with decreasing ranges in colour and airmass.

We can now estimate the error on the zeropoints zp. First, let's make the following settings:

$$\begin{array}{lcl} S_d & = & \sum d_i \, w_i \\ S_c & = & \sum c_i \, w_i \\ S_z & = & \sum z_i \, w_i \end{array} \tag{23}$$

so that Eq. 11 becomes

$$zp = \frac{S_d}{W} - \gamma \bar{c} + k \bar{z} \tag{24}$$

and hence the uncertainty on zp can be estimated as:

$$\sigma_{zp}^2 = \frac{\sigma_{S_d}^2}{W^2} + \bar{c}^2 \sigma_\gamma^2 + \bar{z}^2 \sigma_k^2 \tag{25}$$

where, again, we have assumed that the uncertainties on the standard stars colours and airmasses are negliegeable. The error on S_d is simply given by

$$\sigma_{S_d}^2 = \sum_{i=1}^N \left(\frac{\partial S_d}{\partial d_i}\right)^2 \, \sigma_i^2 = W \tag{26}$$

Hence, the RMS error on zp is given by the following expression:

$$\sigma_{zp}^2 = \frac{1}{W} + \bar{c}^2 \sigma_\gamma^2 + \bar{z}^2 \sigma_k^2 \tag{27}$$

3 A small semplification

The solution we have presented in the previous section assumes that we know the uncertainties σ_i on the single measurements d_i. In practice this might be not the case. For this reason we try to semplify our life assuming that all measurements are affected by the same uncertainty σ. This is actually not too far from reality, since the photometric standard stars commonly used to determine the photometric solution have roughly the same magnitudes.

Following this approach, σ can be estimated *a posteriori*, once the photometric solution has been found, using the following expression:

$$\sigma^2 = \frac{1}{N-3} \sum_{i=1}^{N} \{d_i - (zp + \gamma \times c_i - k \times z_i)\}^2 \tag{28}$$

where $(N-3)$ is the number of degrees of freedom. With this assumption, Eqs. 15 become:

$$\begin{aligned} S_{dc} &= \sum \Delta d_i \Delta c_i \\ S_{dz} &= \sum \Delta d_i \Delta z_i \\ S_{cc} &= \sum \Delta c_i^2 \\ S_{cz} &= \sum \Delta c_i \Delta z_i \\ S_{zz} &= \sum \Delta z_i^2 \end{aligned} \tag{29}$$

and $W = N/\sigma^2$, while Δ is still given by Eq. 17. The colour term and extinction coefficient are found substituting these values in Eqs. 28 and 19. The uncertainties on γ and k are now:

$$\sigma_\gamma^2 = \sigma^2 \, \frac{S_{zz}}{\Delta} \quad ; \quad \sigma_k^2 = \sigma^2 \, \frac{S_{cc}}{\Delta} \tag{30}$$

while the uncertainty on zp is given by the following expression:

$$\sigma_{zp}^2 = \frac{\sigma^2}{N} + \bar{c}^2 \sigma_\gamma^2 + \bar{z}^2 \sigma_k^2 \tag{31}$$

or, more explicitely:

$$\sigma_{zp}^2 = \sigma^2 \left\{ \frac{1}{N} + \frac{\bar{c}^2 S_{zz}}{\Delta} + \frac{\bar{z}^2 S_{cc}}{\Delta} \right\} \tag{32}$$

where $\bar{c} = \sum c_i/N$ and $\bar{z} = \sum z_i/N$.

4 An example

We want now to apply our method to a real example. For this purpose we use the results given by the ESO–FORS Pipeline for the observations of two different Landolt standard fields (PG0231, PG2331) at two different airmasses (cf. Tab. 1).

The original data are plotted in Fig. 1 as empty symbols (lower part of the plot), where both the colour dependency and the extinction effect are visible.

The application of the method discussed here gives $\gamma = (0.02 \pm 0.02)$ and $k_V = (0.19 \pm 0.04)$ mag airmass^{-1}, the estimated measurement error being $\sigma = 0.02$ mag. Substituting these values in Eqs. 12 and 31 we finally have $zp = (27.54 \pm 0.08)$ mag.

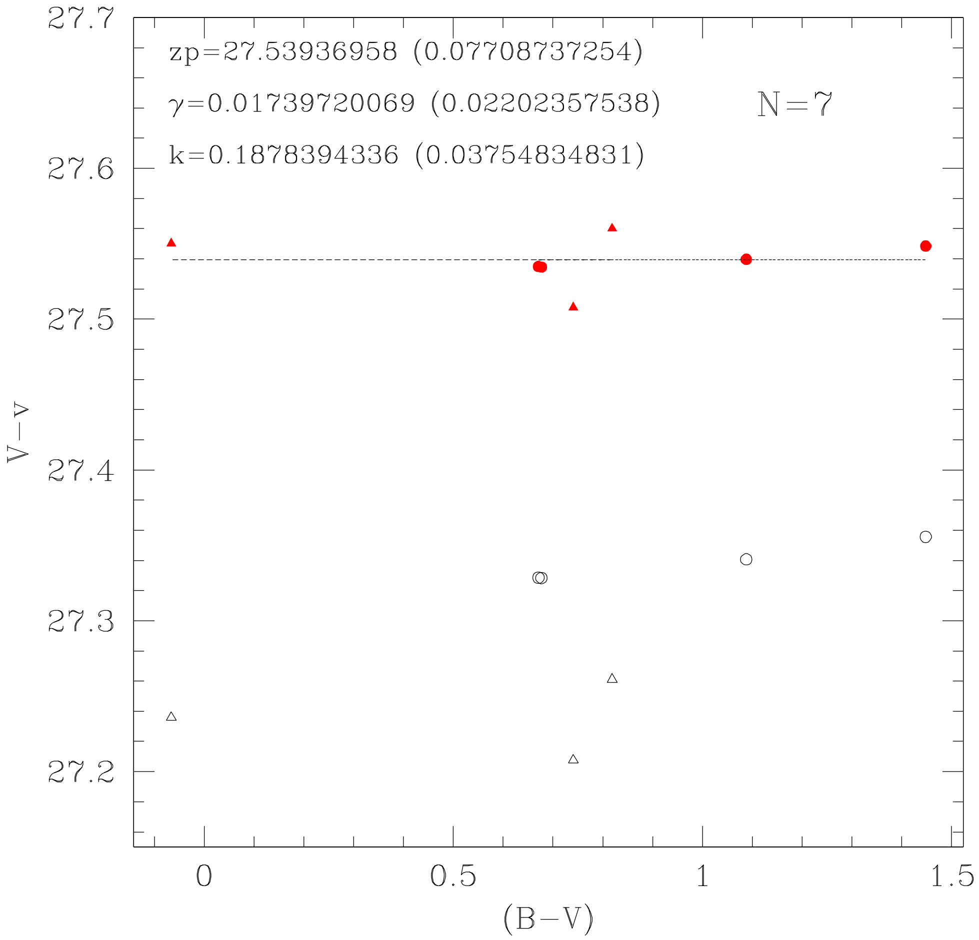

Figure 1: Application of the method to the data of Tab. 1. Empty symbols represent the original data while filled symbols represent the data after correction for extinction and colour term. Triangles are high airmass observations while circles represent low airmass observations.

Star	$(B-V)$	z	$V-v$
PG0231+051C	0.671	1.1601	27.3286
PG0231+051E	0.677	1.1601	27.3284
PG0231+051D	1.088	1.1601	27.3407
PG0231+051B	1.448	1.1601	27.3557
PG2331+055	−0.066	1.6672	27.2359
PG2331+055A	0.741	1.6672	27.2075
PG2331+055B	0.819	1.6672	27.2612

Table 1: relevant data for two different Landolt fields observed with FORS1 on Aug 03, 2000. $V-v$ was computed by the FORS–Pipeline using ADUs.

5 A less general case

Due to the fact that nowadays the photometric standard stars are observed in fields, a good colour coverage is almost always secured. The same does not apply for the airmass coverage. Often the fields are observed at similar airmasses or, even worse, only one field is available. In that case it is clear that $\Delta z_i = 0$ for all stars and hence it is impossible to determine the extinction coefficient. IN this particular case, the Eqs. 16 reduce to the following simple equation:

$$\gamma = \frac{S_{dc}}{S_{cc}} \tag{33}$$

and the uncertainty is then:

$$\sigma_\gamma^2 = \frac{1}{S_{cc}} \tag{34}$$

where S_{cc} and S_{dc} are defined in Eqs. 15. In the case that $\sigma_i = \sigma$, this expression becomes:

$$\sigma_\gamma^2 = \frac{\sigma^2}{S_{cc}} \tag{35}$$

where S_{cc} is given in Eqs. 29 and σ is given by

$$\sigma^2 = \frac{1}{N-2} \sum_{i=1}^{N} \{d_i - (zp + \gamma \times c_i - k \times z_i)\}^2 \tag{36}$$

since there are only two free parameters now.

It is clear that in this case k has to be assumed (with some uncertainty σ_k) and then zp and σ_{zp} can be computed with the expressions we have discussed above.

As an example, we use the data of the second field in Tab. 1 to simulate a case where the airmass range is zero. Imposing $k_V = 0.18$ and $\sigma_k = 0.04$ derived from the previous example, we get $\gamma = 0.00 \pm 0.34$ and $zp = 27.53 \pm 0.20$. These values are indeed very similar to those obtained using the full set of data, but the estimated errors are now much larger, as expected because of the small number of data ($N = 3$) and the limited colour range (cf. Fig. 2). Note that in this case the estimated measurement error is $\sigma = 0.03$, i.e. very similar to that we have obtained using the complete data set.

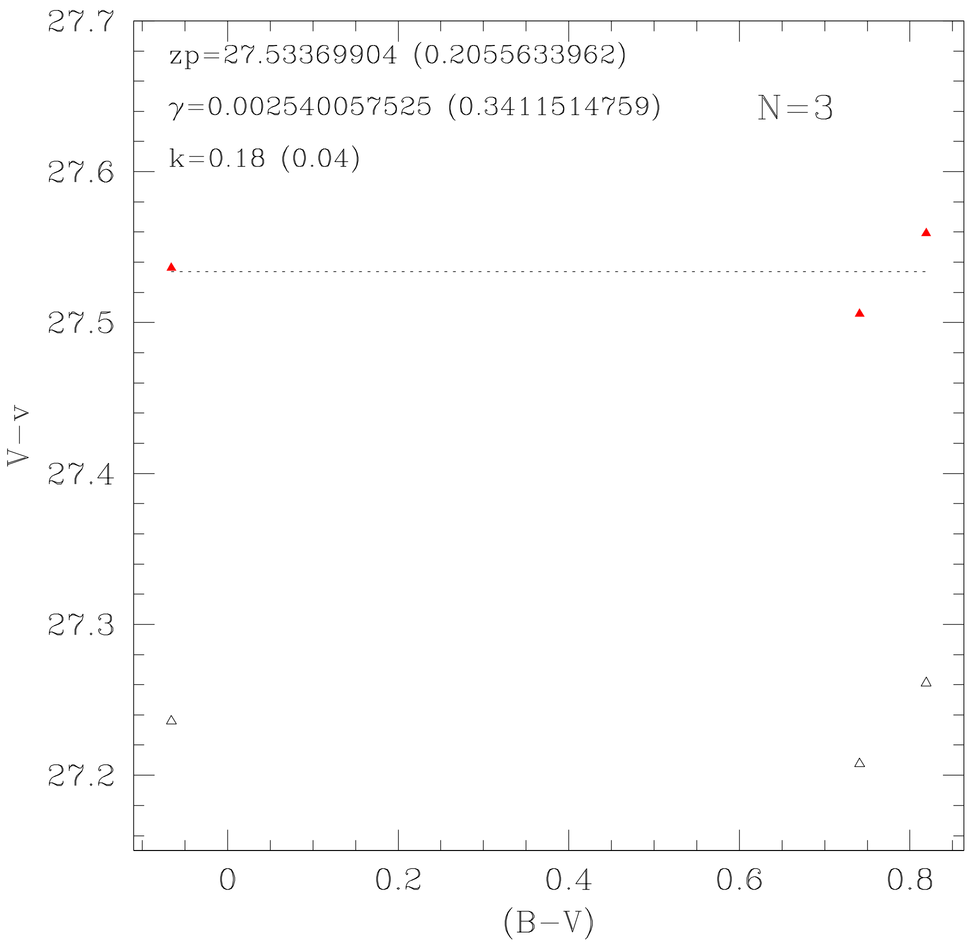

Figure 2: Same as in Fig. 1, with a reduced data set (PG2331 only). See text.

6 References

1. Harris, W.E., Pim Fitzgerald, M. & Cameron Reed, B. 1981, PASP 93, 507