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The Monte Carlo Method in Astrophysics

(Part 3)

Luis A. Aguilar

aguilar@astrosen.unam.mx

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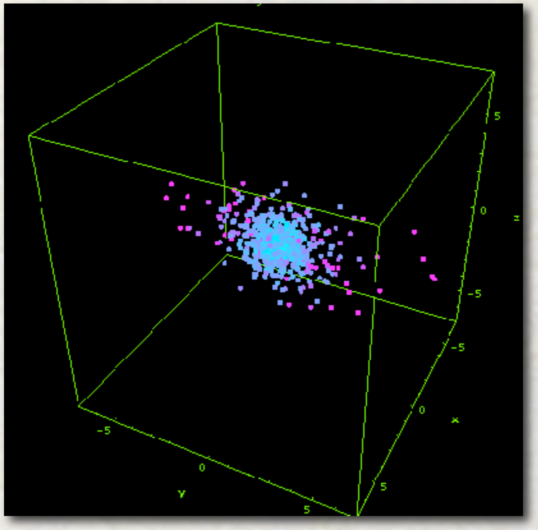
1st Example

Initial conditions for N-body simulations



千里之行始于足下

The longest journey begins with the first step

Ancient Chinese proverb

Random realizations of galactic models

Our 1st example is the generation of random realizations of galactic models.

A dynamical model is described by the phase space density distribution function:

$$f(x_i, v_i; t)$$

This function gives us the fraction of the mass of the model that lies within an element of phase-space volume $d^3x d^3v$ at a given time.

It is obtained solving simultaneously the collisionless Boltzmann and Poisson equations:

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} - \nabla\phi \cdot \frac{\partial f}{\partial \vec{v}} = 0$$

$$\nabla^2 \phi = 4\pi G \rho$$

Random realizations of galactic models

We are, in general, interested in steady state solutions.
Even so, this is a very difficult problem to solve and so few self-consistent models exist.

A result from Galactic Dynamics that is often used to find self-consistent models is the so-called *Jeans Theorem* which says that a time-independent solution depends on the phase-space variables only through the so-called *isolating integrals of motion*:

Any steady-state solution of the collisionless Boltzmann equation depends on the phase-space coordinates only through integrals of motion in the galactic potential, and any function of the integrals yields a steady-state solution of the collisionless Boltzmann equation.

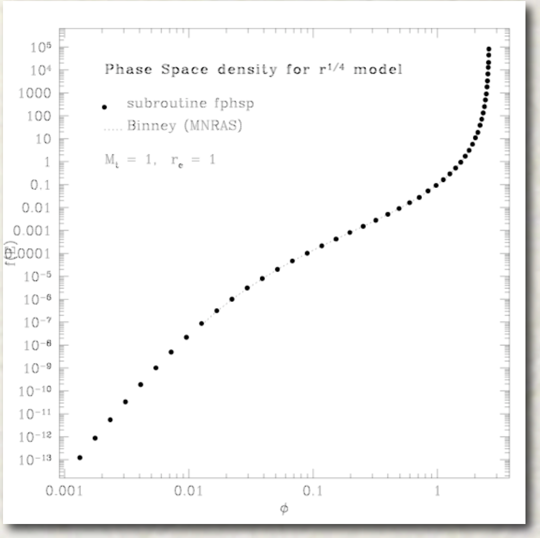
$$\frac{\partial f}{\partial t} = 0 \quad \Rightarrow \quad f(x_i, v_i) = f(I_i(x_i, v_i))$$

Random realizations of galactic models

Jeans Theorem lead to a very popular approach to build galactic models

Decreasing symmetry

f = f(E)	*Spherical isotropic models*
f = f(E,L)	*Spherical anisotropic Models*
F = f(E_J)	*MacLaurin spheroids*



Phase space structure of a de Vaucouleurs galaxy

The addition of integrals of motion lead to ever more complicated and less symmetric models to match galaxies.

Random realizations of galactic models

We will show how to create a random realization of spherical isotropic *f(E)* and spherical anisotropic *f(E,L)* models.

For a spherical isotropic model, the first step is to have the *f(E)* function, or a tabulation of it.

Random realizations of galactic models

We will show how to create a random realization of spherical isotropic *f(E)* and spherical anisotropic *f(E,L)* models.

For a spherical isotropic model, the first step is to have the *f(E)* function, or a tabulation of it.

The next step is to compute the density profile as an integral over velocity space:

$$\rho(\mathbf{r}) = \int f(E(\mathbf{x},\mathbf{v}))\, d^3v$$

Random realizations of galactic models

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We can now create a random realization in configuration space of a spherical model with the density profile we just found.

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We can now create a random realization in configuration space of a spherical model with the density profile we just found.

Our last step is to assign a velocity to each particle.

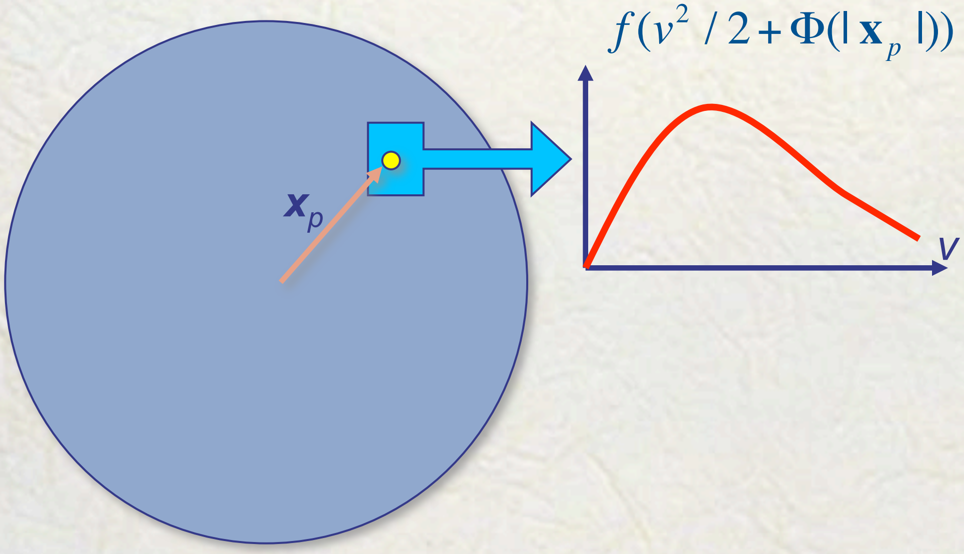
Random realizations of galactic models

How do we pick velocities?

This is done by randomly picking a velocity vector from the corresponding spherical velocity distribution at the position of the particle:

$$f(\mathbf{v}) = f(E(\mathbf{x}_p, \mathbf{v}))$$

$$f(v^2/2 + \Phi(|\mathbf{x}_p|))$$



Random realizations of galactic models

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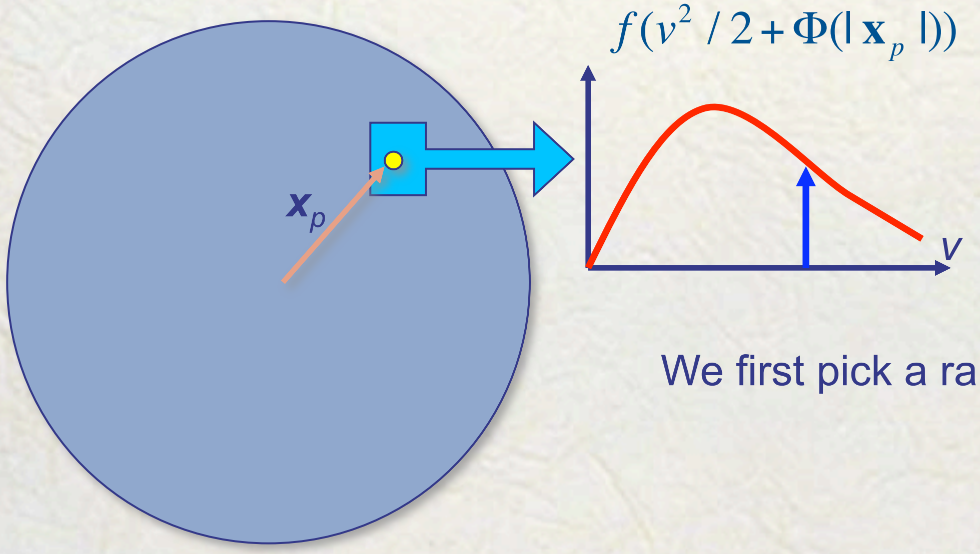

We first pick a random speed

Random realizations of galactic models

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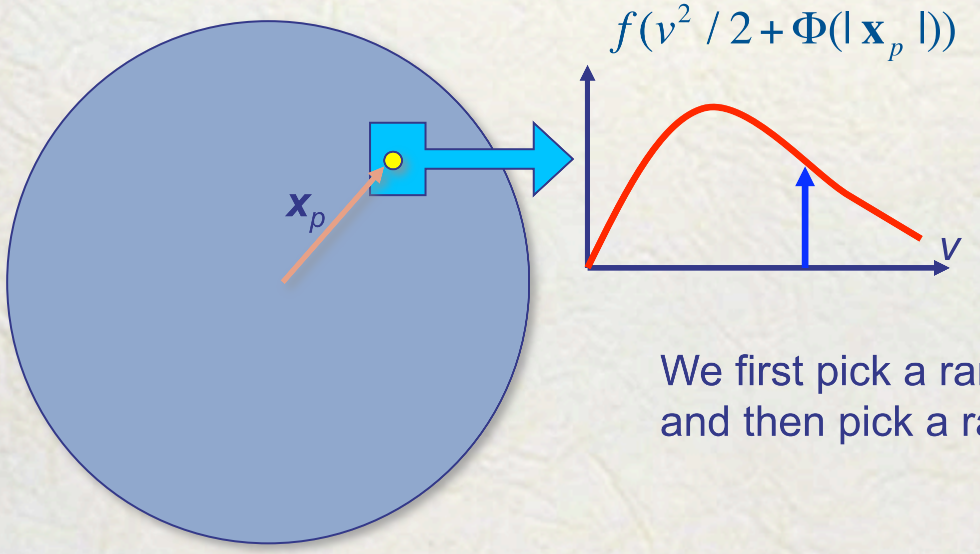
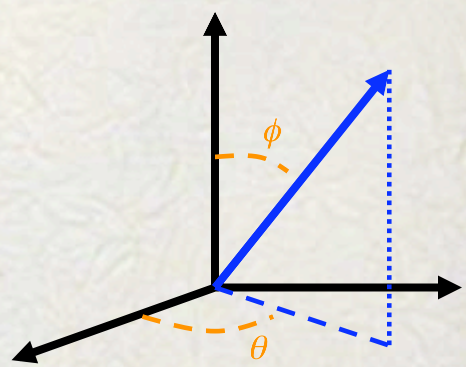




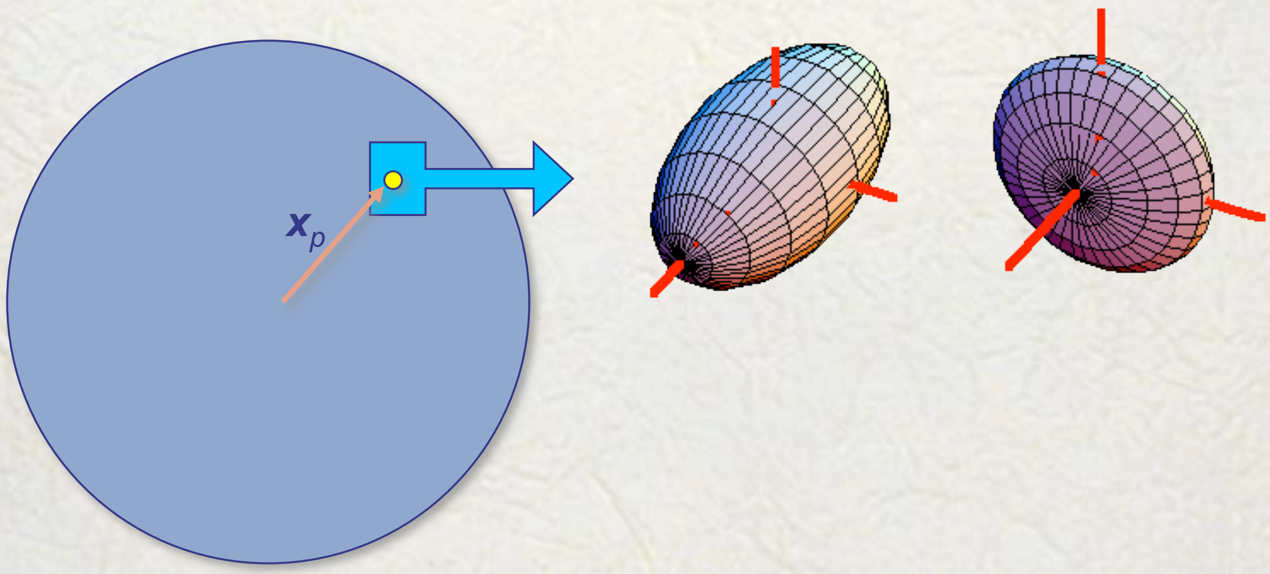
We first pick a random speed,
and then pick a random orientation.

Random realizations of galactic models

In the case of a spherical, anisotropic model, the procedure is the same

In the case of a distribution function of the form *f*(*E*,*L*),
we obtain a bivariate velocity distribution:

$$f(v_r, v_t) = f(E(\mathbf{x}_p, \mathbf{v}), L(\mathbf{x}_p, v_t)) = f((v_r^2 + v_t^2)/2 + \Phi(|\mathbf{x}_p|),\ L(|\mathbf{x}_p| \times v_t))$$



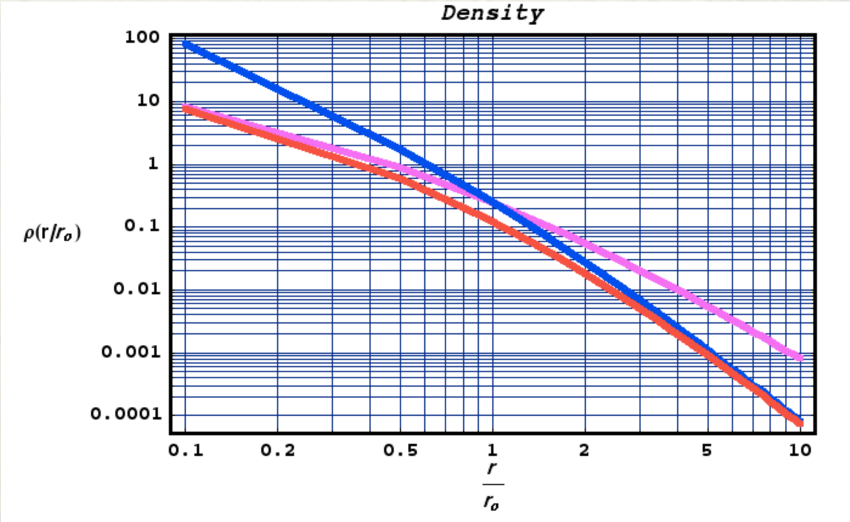
Random realizations of galactic models

There is an extensive literature on spherical models that have been proposed:

Jaffe : $\rho_J(r) = (M/4\pi r_o^3)(r/r_o)^{-2}(1+r/r_o)^{-2}$

Hernquist : $\rho_H(r) = (M/2\pi r_o^3)(r/r_o)^{-1}(1+r/r_o)^{-3}$

NFW : $\rho_{NFW}(r) = \rho_o\,(r/r_o)^{-1}(1+r/r_o)^{-2}$



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Random realizations of galactic models

The distribution function *f(E)* of isotropic models can be obtained, starting from the density profile ρ, using Eddington's formula (e.g. Binney & Tremaine, Galactic Dynamics, chapter 4):

$$f(E) = \frac{1}{\sqrt{8}\pi^2} \frac{d}{dE} \int_0^E \frac{d\phi}{\sqrt{E-\phi}} \frac{d\rho}{d\phi}$$

Jaffe: Jaffe (1983) *MNRAS* **202**, 995
Hernquist: Hernquist (1990) *ApJ* **356**, 359
NFW: Lokas & Mammon (2001) *MNRAS* **321**, 155

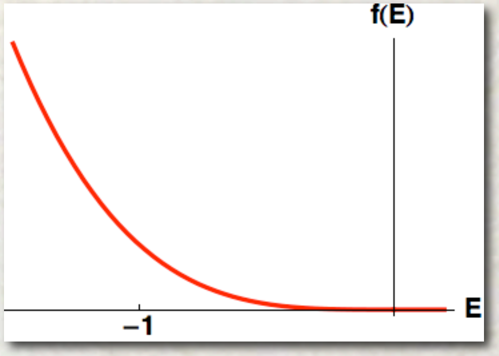
Random realizations of galactic models

The Plummer model revisited

We now come back to the Plummer model to compute the velocities.

The phase space distribution function is given by:

$$f_P(E) = \begin{cases} f_o \mid E \mid^{7/2}, & \phi_o < E < 0, \\ 0, & \text{elsewhere} \end{cases}$$



where,

$$f_o = \frac{3\sqrt{2^7}}{7\pi^3 G^5}\left(\frac{r_o}{M^2}\right)^{7/2}$$

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Random realizations of galactic models

The Plummer model revisited

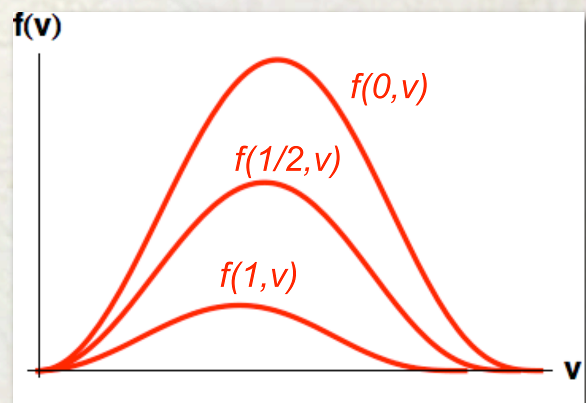
From *f(E)* we can compute the local velocity distribution:

$$f(v;r) = Av^2[B(r) - v^2/2]^{7/2}$$

where,

$$A = 4\pi f_o, \quad B(r) = M/\sqrt{r^2 + r_o^2}$$

We note that, $v_{esc} = \sqrt{2B},$



and the maximum is at:

$$v^* = (2/3)\sqrt{B}, \quad f_{\max} = \frac{4\sqrt{7^7}AB^{9/2}}{19{,}683}$$

Interlude:
Are those groups real?

Before going into our 2nd example, we make an intermission to illustrate the use of the Monte Carlo method in a particular astronomical problem:

Assessing the significance of stellar groups in a survey of high velocity stars in the solar neighborhood.

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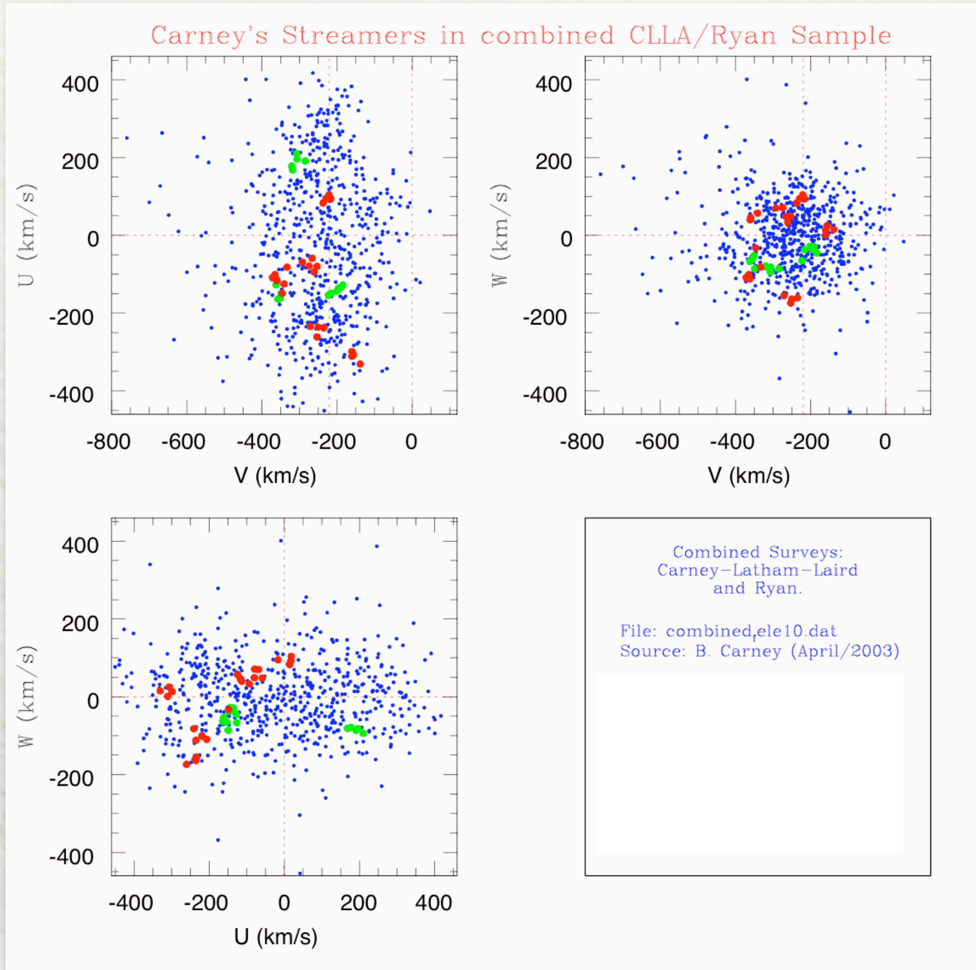
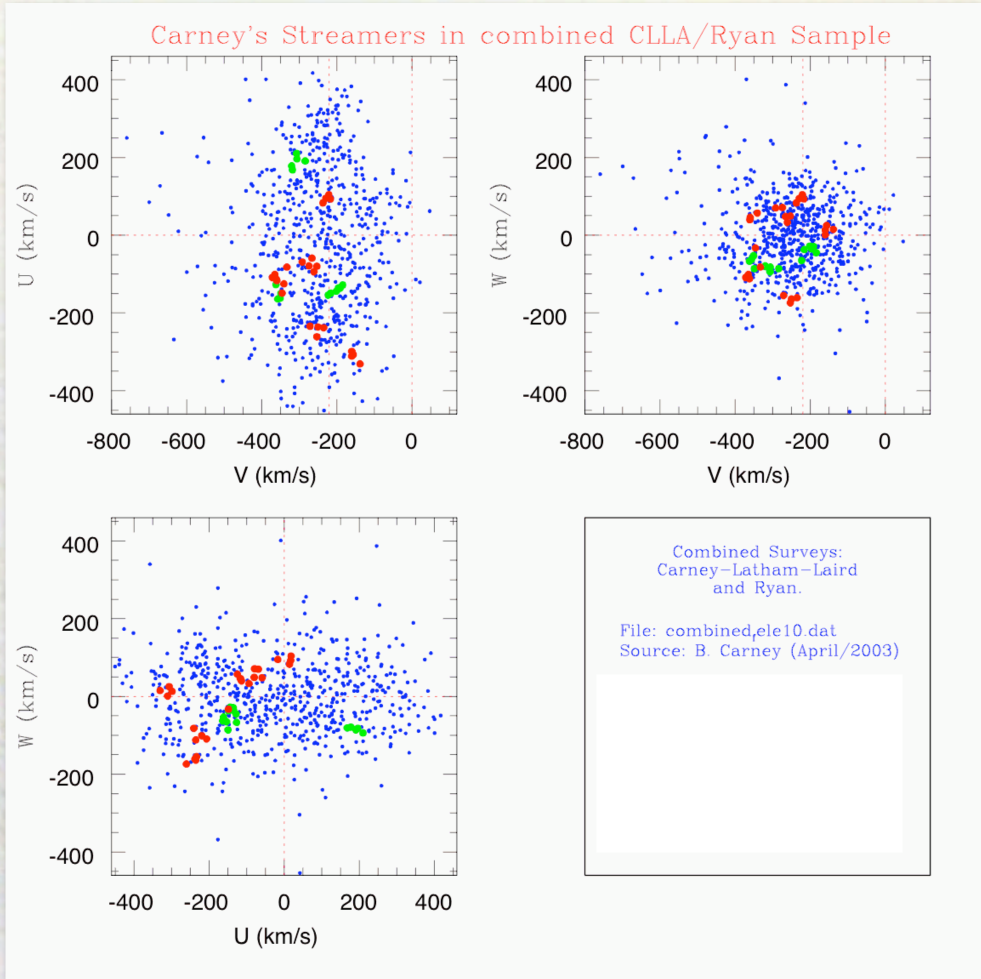
Interlude:
Are those groups real?



The figure shows the velocities of a sample of 692 high-velocity stars in the solar neighborhood (CLLA, 2001).

This survey provides positions, velocities and metallicities for each star.

Interlude:
Are those groups real?



The question was whether there are groups of stars that could be remnants of destroyed stellar groups (moving groups)

These would be groups in Velocity space.

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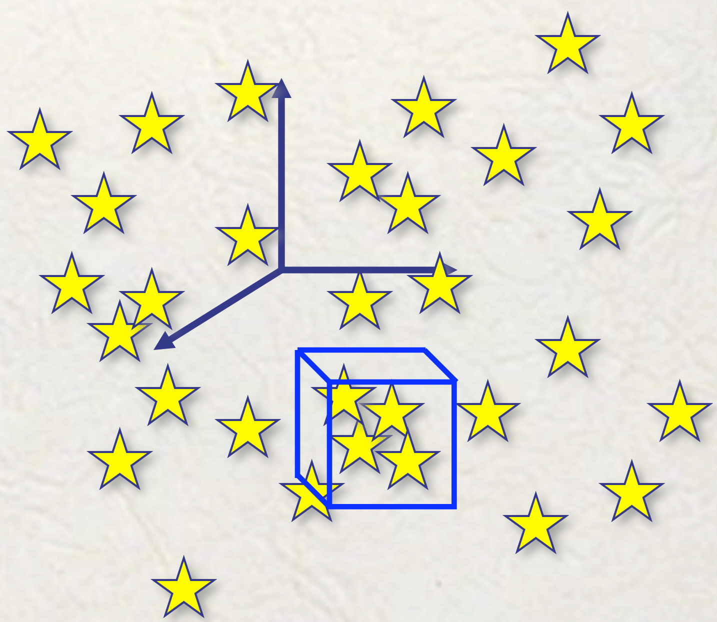
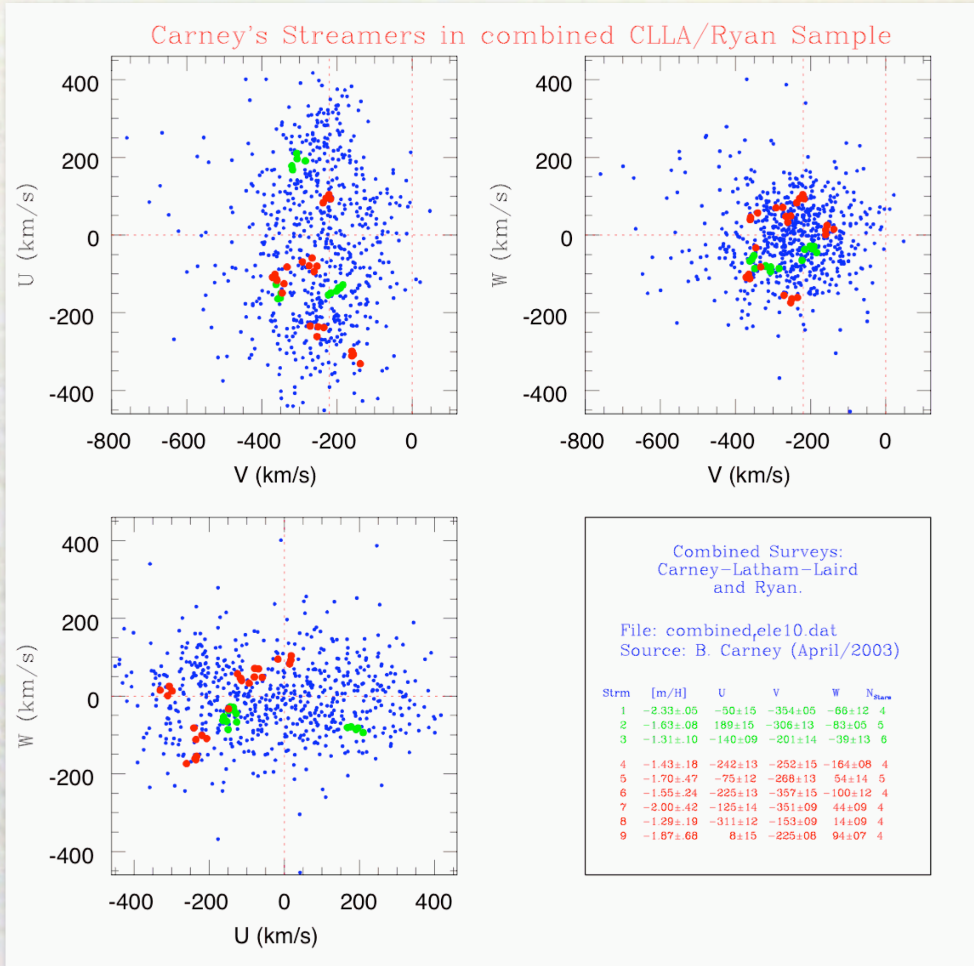
Interlude:
Are those groups real?

First, we need a quantitative definition of a group

At least 4 stars such that their spread in each velocity component is less that 40 *km/s*, and whose velocity dispersion along each direction is less than 15 *km/s*.



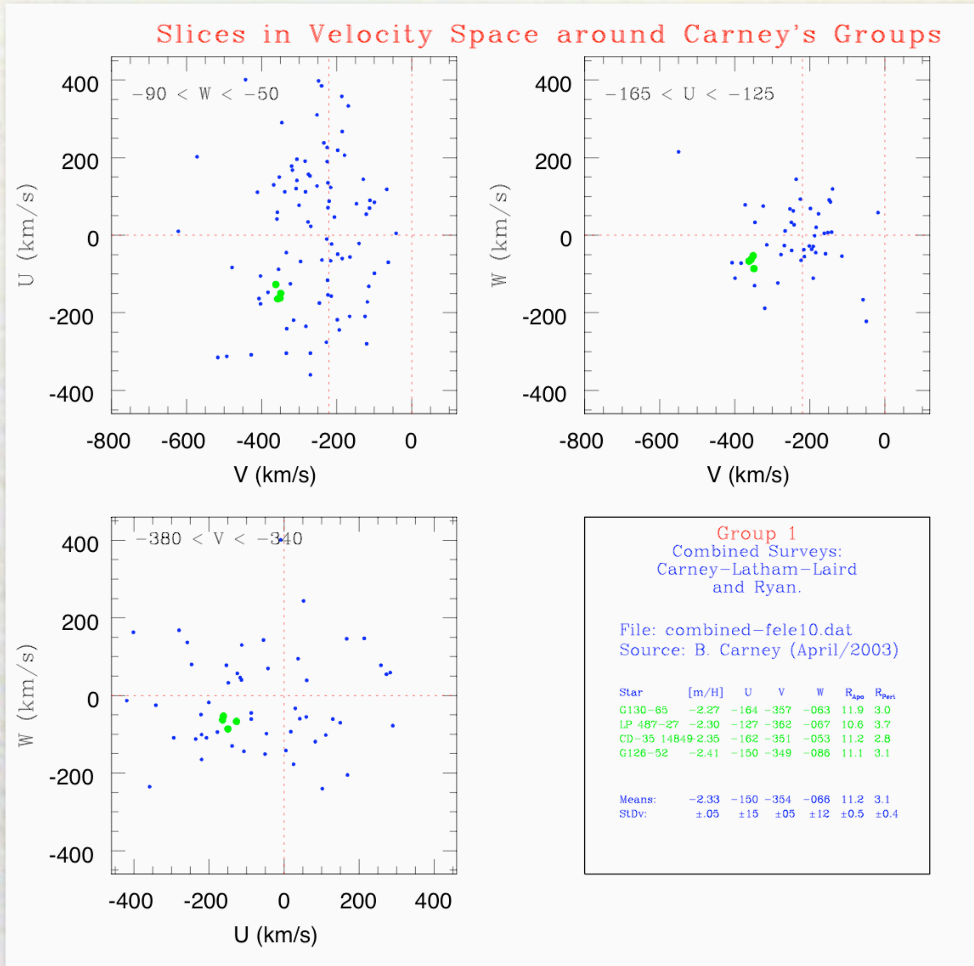
Interlude:
Are those groups real?



Strm	[m/H]	U	V	W	N_{Stars}
1	−2.33±.05	−50±15	−354±05	−66±12	4
2	−1.63±.08	189±15	−306±13	−83±05	5
3	−1.31±.10	−140±09	−201±14	−39±13	6
4	−1.43±.18	−242±13	−252±15	−164±08	4
5	−1.70±.47	−75±12	−268±13	54±14	5
6	−1.55±.24	−225±13	−357±15	−100±12	4
7	−2.00±.42	−125±14	−351±09	44±09	4
8	−1.29±.19	−311±12	−153±09	14±09	4
9	−1.87±.68	8±15	−225±08	94±07	4

Several groups were identified in this manner.

Interlude:
Are those groups real?



Star	[m/H]	U	V	W	R_{Apo}	R_{Peri}
G130−65	−2.27	−164	−357	−063	11.9	3.0
LP 487−27	−2.30	−127	−362	−067	10.6	3.7
CD−35 14849	−2.35	−162	−351	−053	11.2	2.8
G126−52	−2.41	−150	−349	−086	11.1	3.1
Means:	−2.33	−150	−354	−066	11.2	3.1
StDv:	±.05	±15	±05	±12	±0.5	±0.4

They looked like bona fide kinematical groups.

1st group: 4 stars

U: *-150 ± 15 km/s*
V: *-354 ± 05 km/s*
W: *-066 ± 12 km/s*
[Fe/H]: -2.33 ± .05

Slices in velocity space

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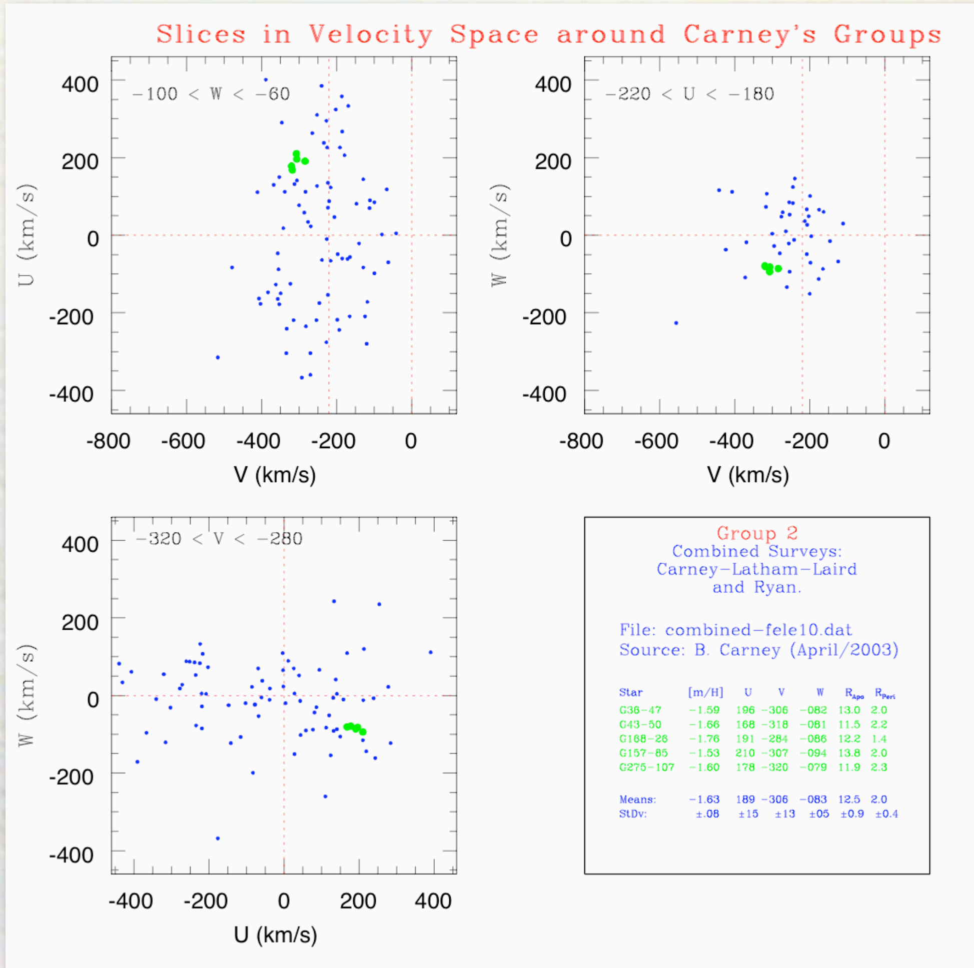
Interlude:
Are those groups real?



Star	[m/H]	U	V	W	R_{Apo}	R_{Peri}
G36−47	−1.59	196	−306	−082	13.0	2.0
G43−50	−1.66	168	−318	−081	11.5	2.2
G168−26	−1.76	191	−284	−086	12.2	1.4
G157−85	−1.53	210	−307	−094	13.8	2.0
G275−107	−1.60	178	−320	−079	11.9	2.3
Means:	−1.63	189	−306	−083	12.5	2.0
StDv:	±.08	±15	±13	±05	±0.9	±0.4

They looked like bona fide kinematical groups.

2nd group: 5 stars

U: -189 ± 15 km/s
V: -306 ± 13 km/s
W: -083 ± 05 km/s
[Fe/H]: -1.63 ± .08

Slices in velocity space

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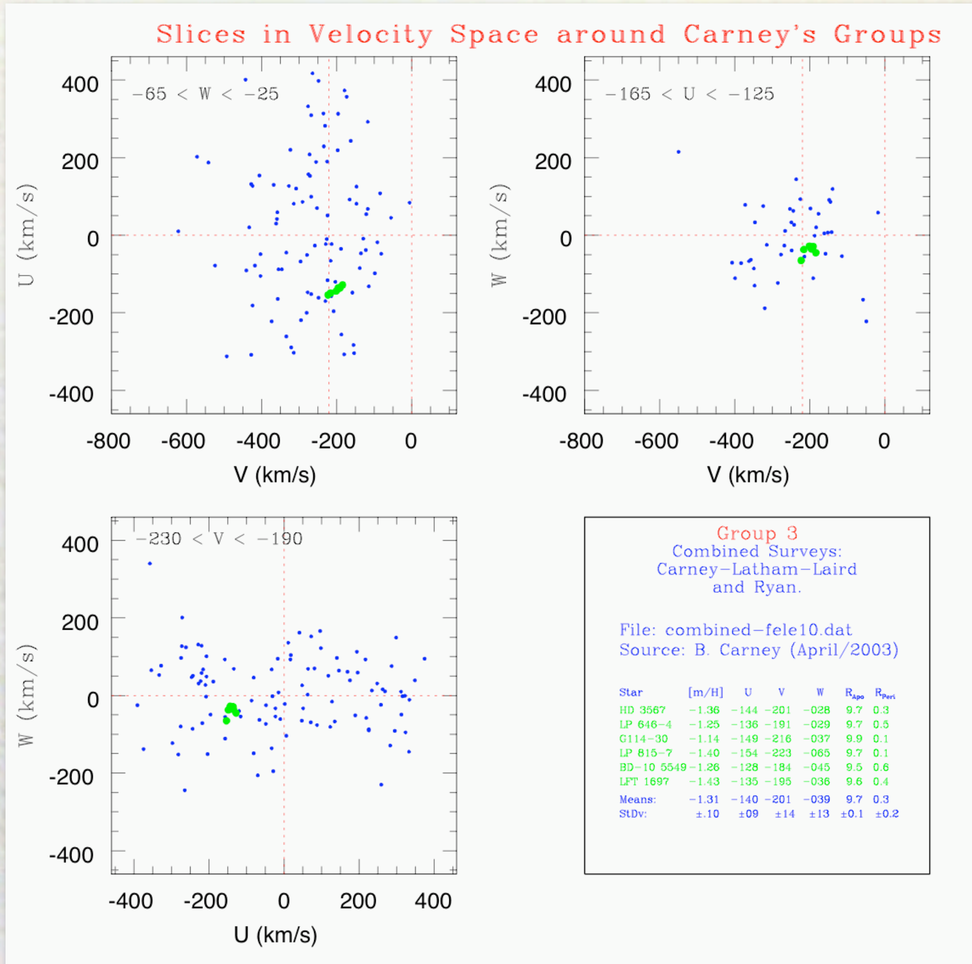
Interlude:
Are those groups real?



Star	[m/H]	U	V	W	R_{Apo}	R_{Peri}
HD 3567	−1.36	−144	−201	−028	9.7	0.3
LP 646−4	−1.25	−136	−191	−029	9.7	0.5
G114−30	−1.14	−149	−216	−037	9.9	0.1
LP 815−7	−1.40	−154	−223	−065	9.7	0.1
BD−10 5549	−1.26	−128	−184	−045	9.5	0.6
LFT 1697	−1.43	−135	−195	−036	9.6	0.4
Means:	−1.31	−140	−201	−039	9.7	0.3
StDv:	±.10	±09	±14	±13	±0.1	±0.2

They looked like bona fide kinematical groups.

3rd group: 6 stars

U: -140 ± 09 km/s
V: -201 ± 14 km/s
W: -039 ± 13 km/s
[Fe/H]: -1.31 ± .10

… and some others.

Slices in velocity space

Interlude:
Are those groups real?

Have we found evidence of rich structure in this sample?

Remember, these are tight kinematical groups
and there are several of them

But what if they are just chance clusterings?

How can we be sure?
Do a Monte Carlo!

Interlude: Are those groups real?

The null hypothesis is that these are just chance fluctuations in a random draw of 692 stars from a smooth velocity distribution for halo stars in the solar neighborhood.

So, our 1st step is to find a plausible smooth kinematical model

We used a velocity ellipsoid with the following dispersions:

$\sigma_U = 153$ *km/s*, $\sigma_V = 93$ *km/s*, $\sigma_W = 107$ *km/s*,

Sommer-Larsen (1999) and $V_{LSR} = 220$ *km/s.*

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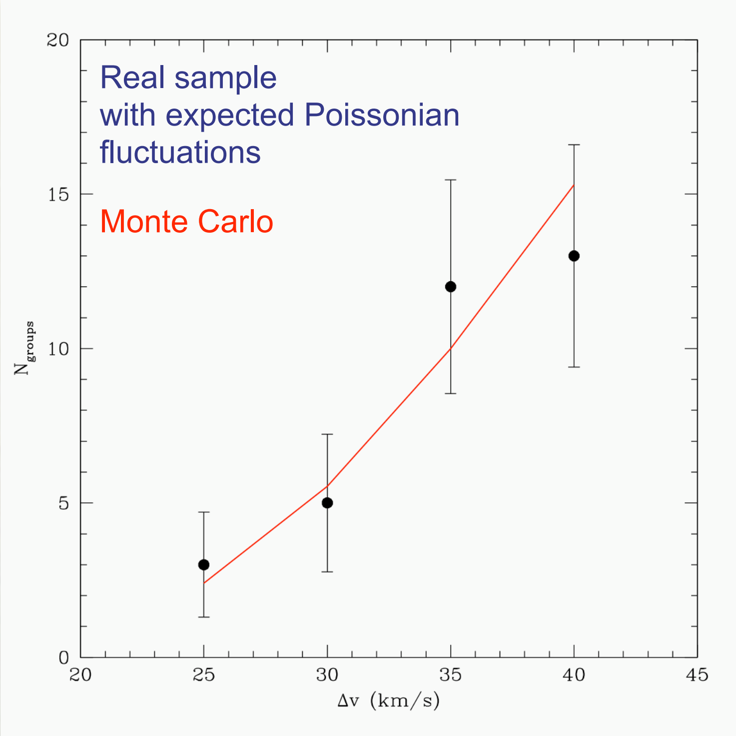
Interlude:
Are those groups real?

Then we obtained 10^4 random realizations of the CLLA survey, and applied our group definition to each one, varying the tightness in velocity space.

Table 1

Number of Kinematical Groups

σ_v	Real Sample	Monte Carlo
40	13 ± 3.6	15.3
35	12 ± 3.46	10.0
30	5 ± 1.7	5.54
25	3 ± 1.7	2.397



As can be seen, all groups, even the jump in going from σ_v<30 to *35 km/s*, are consistent with being just random fluctuations.

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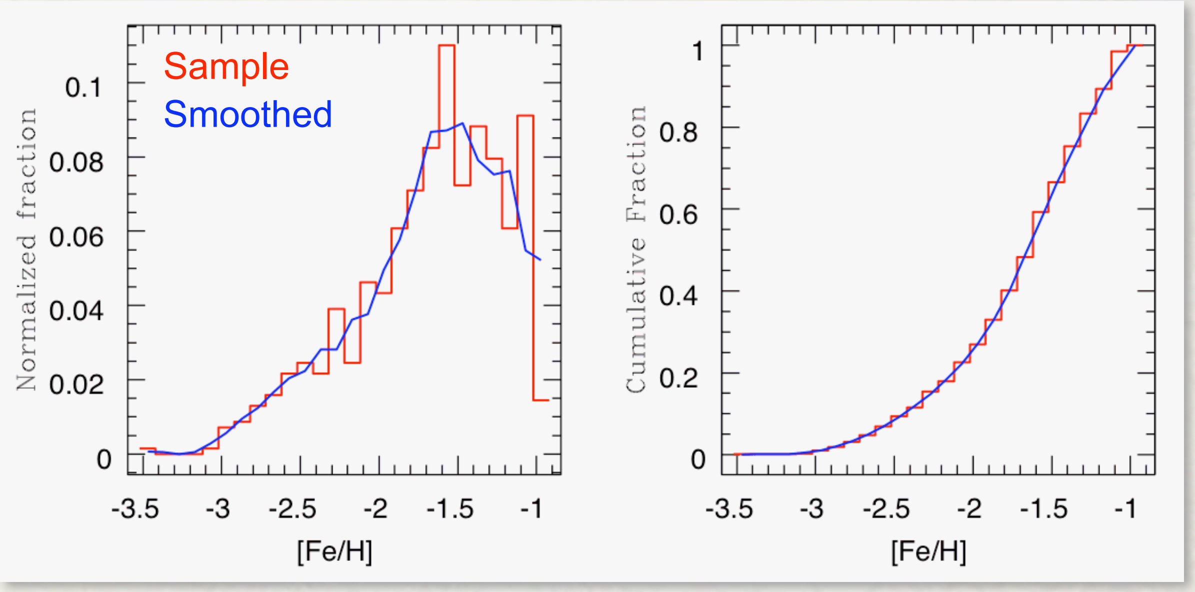
Interlude:
Are those groups real?

But, what about the additional clumping in metallicity?



In this case we had to build our "null hypothesis" for a metallicity distribution without clumps by smoothing the metallicity histogram from the real sample.

Interlude: Are those groups real?

We searched for groups varying the tightness in velocity and metallicity

Table 2

Metallicity Subgroups with $N_* = 4$

	$\Delta[Fe/H] \leq$							
$\Delta v \leq$	0.05	0.10	0.15	0.20	0.30	0.40	0.50	0.60
40 km/s	**0** (0.12)	**0** (0.71)	**2** (1.6)	**2** (2.6)	**3** (4.1)	**5** (5.1)	**6** (5.9)	**10** (6.5)
35	**0** (0.08)	**0** (0.43)	**0** (1.0)	**1** (1.6)	**2** (2.6)	**4** (3.2)	**4** (3.8)	**8** (4.2)
30	**0** (0.03)	**0** (0.20)	**0** (0.47)	**1** (0.77)	**1** (1.2)	**1** (1.5)	**1** (1.8)	**2** (2.1)
25	**0** (0.01)	**0** (0.08)	**0** (0.19)	**1** (0.30)	**1** (0.46)	**1** (0.61)	**1** (0.75)	**2** (0.87)

Number of groups found in real sample.

(Mean number of groups found in 10^4 random realizations).

In general, the number of groups observed in the real sample, is consistent with the numbers from the null hypothesis.

Interlude:
Are those groups real?

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40 km/s	**0** (0.12)	**0** (0.71)	**2** (1.6)	**2** (2.6)	**3** (4.1)	**5** (5.1)	**6** (5.9)	**10** (6.5)
35	**0** (0.08)	**0** (0.43)	**0** (1.0)	**1** (1.6)	**2** (2.6)	**4** (3.2)	**4** (3.8)	**8** (4.2)
30	**0** (0.03)	**0** (0.20)	**0** (0.47)	**1** (0.77)	**1** (1.2)	**1** (1.5)	**1** (1.8)	**2** (2.1)
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Number of groups found in real sample.

(Mean number of groups found in 10^4 random realizations).

In general, the number of groups observed in the real sample, is consistent with the numbers from the null hypothesis. However, there seems to be one significative group: if we scan the last row from right to left, we see that, while the expected Monte Carlo number goes down, there is one group that persists until Δ*[Fe/H]=0.20;* its presence is a factor of 3 above the expected number.

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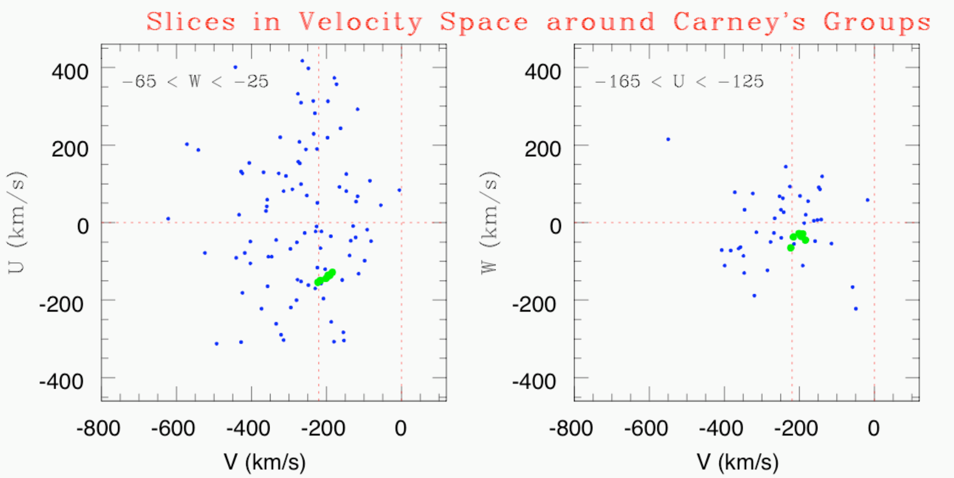
Interlude:
Are those groups real?



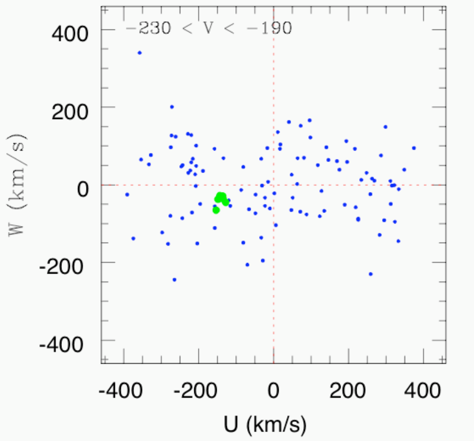


Table 3

Candidate streamer

#	Name	[m/H]	U	V	W
8	HD 3567	-1.36	-144	-201	-28
416	LFT 1697	-1.43	-135	-195	-36
455	G46–004	-1.25	-136	-191	-29
642	815–007	-1.40	-154	-223	-65
652	−10 5549	-1.26	-128	-184	-45

The kinematics of this group corresponds to a halo object with substantial velocity along the galactocentric direction and almost no rotation with respect to an inertial observer.
The resulting orbit has an apogalacticon of *~9.5 kpc* and a perigalacticon of only *~0.5 kpc*.
The eccentricity is *~0.9* and the maximum height over the galactic plane is *~5.7 kpc*.

This roughly coincides with the orbit of the globular cluster ω-Cen.

Interlude:
Are those groups real?

So, what is the lesson?

We seemed to have identified a substantial number of groups in the CLLA sample.

Using Monte Carlo simulations of the survey, we were able to assess the statistical significance of them and discovered that, only one of them, seems to be significative.

Monte Carlo helped us to separate the illusory from the real

Homework

3rd Assignment

- Extend the program of the 2nd assignment to include the generation of velocities.
- Use the phase space distribution function of the Plummer model to find the local velocity distribution at a given radial position.
  Use it to draw random speeds for each particle.
- Use the routine that generates random positions in the unit sphere to convert speeds in velocities.
- Compute the total kinetic and potential energies of the final random realization and verify the the virial ratio ($2K/|W|$) is one.

Note: Assume that $G=1$, $r_o=1$ and $M=1$.

End of third lecture