IV Latin American Workshop on Plasma Physics
Buenos Aires - Argentina
16-27 July, 1990

# Analytical Rappresentation of the Dielectric Tensor for a Relativistic Anisotropic Plasma with Loss-cone Distribution Function

P.De Chiara, G.Chiozzi*

Physics Departement, University of Ferrara, 44100 Ferrara, Italy

* Present address: IBM Centro Ricerche, Milano, Italy

# Analytical Rappresentation of the Dielectric Tensor for a Relativistic Anisotropic Plasma with Loss–cone Distribution Function

P. De Chiara, G. Chiozzi*
Physics Departement, University of Ferrara, 44100 Ferrara, Italy

Analytical expressions for the weakly relativistic dielectric tensor components, $\varepsilon_{i,j}$, near the electron cyclotron frequency and harmonics are given for a loss-cone equilibrium distribution function $f_\circ$ with temperature anisotropy. The dielectric tensor is given in the BBW[1] [Baldwin, Bernstein, Weenink] and Bekefi[2] formulations. As considered by Tamor[3], the relevant difference between [1] and [2] appears for anisotropic distributions, when the hermitian property, respected in BBW dielectric tensor harmonic term by term, is obeyed by Bekefi formula only in the global sense, i.e when summed over harmonics.

## I. Introduction

The (relativistic)dielectric tensor $\varepsilon_{i,j}$ relevant to study the interaction between magnetized plasma and electromagnetic waves, (electrostatic approximation is considered), is characterized by an integration over the momentum variables, e.g. $p_\perp$ and $p_\parallel$ (the momentum perpendicular and parallel to the external magnetic field $\underline{B}_\circ = \hat{\underline{z}} B_\circ$, a time integration and a infinite sum over the harmonic number of terms containing the product of two Bessel functions whose argument depends on $p_\perp$ [Bornatici and co-workers[4]]. In the reference frame where the wave vector is $\underline{k} = k_\perp \hat{\underline{x}} + k_\parallel \hat{\underline{z}}$, the BBW[1] relativistic dielectric tensor (only the electron component is considered) is given by:

$$\varepsilon_{i,j} - \delta_{i,j} = 2\pi \left[\frac{\omega_p}{\omega}\right]^2 \sum_{n=-\infty}^{+\infty} (-i) \int_0^{+\infty} dt \int dp_\perp dp_\parallel p_\perp^2 e^{it[\gamma - N_\parallel \frac{p_\parallel}{mc} - n\frac{\omega_c}{\omega}]} [V_i^{(n)}]^* V_j^{(n)} \hat{U} f_\circ$$

$$+ 2\pi \left[\frac{\omega_p}{\omega}\right]^2 \delta_{i,z} \sum_{n=-\infty}^{+\infty} \int dp_\perp dp_\parallel \frac{p_\perp}{\gamma} V_j^{(n)} J_n \hat{F} f_\circ \tag{1}$$

where:

$$\underline{V}^{(n)} \stackrel{\text{def}}{=} \left[\frac{n}{a} J_n(a), -iJ_n'(a), \frac{p_\parallel}{p_\perp} J_n(a)\right] \tag{2}$$

$$\hat{U} \stackrel{\text{def}}{=} \frac{\partial}{\partial p_\perp} + \frac{N_\parallel}{\gamma mc} \hat{F}, \quad \text{with} \quad \hat{F} \stackrel{\text{def}}{=} p_\perp \frac{\partial}{\partial p_\parallel} - p_\parallel \frac{\partial}{\partial p_\perp}. \tag{3}$$

$J_n(a)$ is the Bessel function of the first kind of order $n$ (the harmonic number) and $J_n'$ its derivation with respect to the argument $a = N_\perp(\omega/\omega_c) p_\perp/mc$; $[\underline{V}^{(n)}]^*$ is the complex conjugation of $\underline{V}^{(n)}$; $\gamma$ is the relativistic factor $= \sqrt{1 + (p/mc)^2}$; $N_\perp$ ($N_\parallel$) is the wave refractive index perpendicular (parallel) to $\underline{B}_\circ$ , $\omega_p \equiv (4\pi n_e e^2/m_e)^{\frac{1}{2}}$ and $\omega_c \equiv |eB/m_e c|$ are the plasma and the cyclotron electron frequency respectively.

* Present address: IBM Centro Ricerche, Milano, Italy

In the Bekefi formulation the second line of Eq.(1) appears as it follows:

$$+2\pi\left[\frac{\omega_p}{\omega}\right]^2\sum_{n=-\infty}^{+\infty}(-i)\int_0^{+\infty}dt\int dp_\perp dp_\| p_\perp^2 e^{it[\gamma-N_\|\frac{p_\|}{mc}-n\frac{\omega_c}{\omega}]}[V_i^{(n)}]^* W_j^{(n)} \tag{4}$$

where:

$$\underline{W}^{(n)} \stackrel{\text{def}}{=} \left[0,0,\frac{1}{\gamma p_\|}\left(\gamma-N_\|\frac{p_\|}{mc}-n\frac{\omega_c}{\omega}\right)\hat{F}f_\circ V_z^{(n)}\right]. \tag{5}$$

From the eq.(5) we note that all the dielectric tensor components with an index $z$ are modified.

## II. Dielectric tensor with loss-cone distribution function

It is possible to calculate the dielectric tensor for the loss-cone distribution function of any order $l$ (including temperature anisotropy)

$$f_\circ(p_\perp,p_\|)=\frac{1}{\pi^{3/2}l!(2mT_\|)^{1/2}2mT_\perp}\left[\frac{p_\perp^2}{2mT_\perp}\right]^l\exp\left(-\frac{p_\|^2}{2mT_\|}-\frac{p_\perp^2}{2mT_\perp}\right) \tag{6}$$

by applying to the dielectric tensor for a bi-maxwellian distribution function the following differential operator ($T_\|$ and $T_\perp$ are the parallel and perpendicular temperature respectively):

$$\frac{(-1)^l}{l!}\frac{1}{T_\perp^{l+1}}\frac{\partial^l}{\partial(T_\perp^{-1})^l}(T_\perp). \tag{7}$$

In the weakly relativistic approximation, $\gamma\approx 1+\frac{1}{2}(p_\perp/mc)^2+\frac{1}{2}(p_\|/mc)^2$, by suitably applying Eq.(6) and by using the series rappresentations of $J_n^2$, $J_nJ_n'$ and $J_n'^2$ (the only Bessel function combinations that appear in the dielectric tensor structure), the BBW formulation of the dielectric tensor for loss-cone distribution function reads as it follows:

$$\begin{aligned}\begin{pmatrix}\varepsilon_{xx}-1\\ \varepsilon_{xy}=-\varepsilon_{yx}\\ \varepsilon_{yy}-1\end{pmatrix} = &-\left[\frac{\omega_p}{\omega}\right]^2\frac{mc^2}{T_\|}\sum_{n=1}^{+\infty}\frac{\lambda_\perp^{n-1}}{2^n n!}\sum_{k=0}^{+\infty}c_{n,k}\lambda_\perp^k\frac{(n+k+l)!}{(n+k)!l!}\begin{pmatrix}n^2\\ -in(n+k)\\ A_{n,k}\end{pmatrix}\\ &\times\left\{\Delta^{\left[\substack{+\\ -\\ +}\right]}_{\frac{1}{2},n+k+l}(l)+N_\|^2\frac{\partial}{\partial z_\|}\Delta^{\left[\substack{+\\ -\\ -}\right]}_{\frac{3}{2},n+k+l}(l;\frac{T_\perp}{T_\|})\right\}+\\ &-\left[\frac{\omega_p}{\omega}\right]^2\frac{mc^2}{T_\|}\sum_{k=2}^{+\infty}c_{0,k}\lambda_\perp^{k-1}\frac{(k+l)!}{k!l!}\begin{pmatrix}0\\ 0\\ A_{0,k}\end{pmatrix}\\ &\times\left\{\Delta^{[0]}_{\frac{1}{2},k+l}(l)+N_\|^2\frac{\partial}{\partial z_\|}\Delta^{[0]}_{\frac{3}{2},k+l}(l;\frac{T_\perp}{T_\|})\right\}\end{aligned} \tag{8}$$

$$\begin{pmatrix} \varepsilon_{xz} = \varepsilon_{zx} \\ \varepsilon_{yz} = -\varepsilon_{zy} \end{pmatrix} = + \frac{\bar{\omega}_p^2}{\bar{\omega}} N_\parallel N_\perp \sum_{n=1}^{+\infty} \frac{\lambda_\perp^{n-1}}{2^n n!} \sum_{k=0}^{+\infty} c_{n,k} \lambda_\perp^k \frac{(n+k+l)!}{(n+k)!l!} \begin{pmatrix} n \\ i(n+k) \end{pmatrix}$$
$$\times \left\{ \frac{mc^2}{T_\parallel} \frac{\partial}{\partial z_\parallel} \Delta^{[\pm]}_{\frac{3}{2},n+k+l}(l) + \frac{\partial}{\partial N_\parallel} \left( N_\parallel \Delta^{[\mp]}_{\frac{3}{2},n+k+l}(l; \frac{T_\perp}{T_\parallel}) \right) \right\} +$$
$$- \frac{\bar{\omega}_p^2}{\bar{\omega}} N_\parallel N_\perp \sum_{k=1}^{+\infty} c_{0,k} \lambda_\perp^{k-1} \frac{(k+l)!}{k!l!} \begin{pmatrix} 0 \\ -ik \end{pmatrix}$$
$$\times \left\{ \frac{mc^2}{T_\parallel} \frac{\partial}{\partial z_\parallel} \Delta^{[0]}_{\frac{3}{2},k+l}(l) + \frac{\partial}{\partial N_\parallel} \left( N_\parallel \Delta^{[0]}_{\frac{3}{2},k+l}(l; \frac{T_\perp}{T_\parallel}) \right) \right\} \tag{9}$$

$$\varepsilon_{zz} - 1 = - \left[\frac{\omega_p}{\omega}\right]^2 \left[1 - \frac{T_\parallel}{T_\perp}\delta_{l,0}\right] - \left[\frac{\omega_p}{\omega}\right]^2 \frac{mc^2}{T_\perp} \sum_{n=1}^{+\infty} \frac{\lambda_\perp^n}{2^n n!} \sum_{k=0}^{+\infty} c_{n,k} \lambda_\perp^k \frac{(n+k+l)!}{(n+k)!l!}$$
$$\times \left\{ \frac{\partial}{\partial N_\parallel} \left( N_\parallel \Delta^{[+]}_{\frac{3}{2},n+k+l}(l) \right) + N_\parallel^2 \frac{\partial}{\partial z_\parallel} \left[ 2 + \frac{\partial}{\partial N_\parallel}(N_\parallel) \right] \Delta^{[-]}_{\frac{5}{2},n+k+l}(l; \frac{T_\perp}{T_\parallel}) \right\} +$$
$$- \left[\frac{\omega_p}{\omega}\right]^2 \frac{mc^2}{T_\perp} \sum_{k=0}^{+\infty} c_{0,k} \lambda_\perp^k \frac{(k+l)!}{k!l!}$$
$$\times \left\{ \frac{\partial}{\partial N_\parallel} \left( N_\parallel \Delta^{[0]}_{\frac{3}{2},k+l}(l) \right) + N_\parallel^2 \frac{\partial}{\partial z_\parallel} \left[ 2 + \frac{\partial}{\partial N_\parallel}(N_\parallel) \right] \Delta^{[0]}_{\frac{5}{2},k+l}(l; \frac{T_\perp}{T_\parallel}) \right\} \tag{10}$$

where the $\delta_{l,0}$ recovers the $l = 0$ (bi-maxwellian) system, $z_\parallel \equiv (1 - n\frac{\omega_c}{\omega})mc^2/T_\parallel$, $\lambda_\perp \equiv N_\perp^2 \times [\frac{\omega}{\omega_c}]^2 T_\perp/mc^2$, (the bar defines normalization by $\omega_c$), and:

$$A_{n,k} \equiv \frac{[(2n+k)(2n+k-1) + k(k+1)](n+k) + 2k(2n+k)(n+k-1)}{2[2(n+k)-1]}$$
$$c_{n,k} \equiv \frac{(-1)^k n![2(n+k)!]}{2^k (n+k)!(2n+k)!k!} \tag{11}$$

$$\Delta^{(\pm,0)}_{q,m+l}(l, \frac{T_\perp}{T_\parallel}) \equiv (1 - \frac{T_\perp}{T_\parallel}) W^{(\pm,0)}_{q,m+l+1} - \frac{l}{l+m} W^{(\pm,)0}_{q,m+l}$$
$$\Delta^{(\pm,0)}_{q,m+l}(l) \equiv W^{(\pm,0)}_{q,m+l+1} - \frac{l}{l+m} W^{(\pm,0)}_{q,m+l} \tag{12}$$

The $W^{(\pm,0)}_{q,p}$ are combinations of the genaralized (to temperature anisotropy) Shkarofsky[5] dispersion functions $W_{q,p}(z_\parallel, N_\parallel mc^2/2T_\parallel)$:

$$W_{q,p} = (-i) \int_0^{+\infty} d\tau \frac{e^{i\tau z_\parallel - N_\parallel^2 \frac{mc^2}{2T_\parallel} \frac{\tau^2}{(1-i\tau)}}}{(1-i\tau)^q (1 - i\tau \frac{T_\perp}{T_\parallel})^p} \tag{13}$$

as it follows:

$$W^{(\pm,0)}_{q,p} \equiv W_{q,p}(z_{\parallel(+n,0)}, N_\parallel \frac{mc^2}{2T_\parallel}) \pm (1 \mp \delta_{n,0}) W_{q,p}(z_{\parallel(-n,0)}), N_\parallel \frac{mc^2}{2T_\parallel}). \tag{14}$$

As far as the Bekefi formulation is concerned, that is the proper one to consider any anisotropic system,( only in the non relativistic, $N_{\parallel}^2 \frac{mc^2}{T_{\parallel}} \gg$ [5], asymptotic limit the BBW formulae work for very strog anisotropy,e.g. $\frac{T_{\parallel}}{T_{\perp}} \gg$) Eqs.(9),(10) are so modified:

$$\begin{pmatrix} \varepsilon_{xz} = \varepsilon_{zx} \\ \varepsilon_{yz} = -\varepsilon_{zy} \end{pmatrix} = +\frac{\bar{\omega}_p^2}{\bar{\omega}} N_{\parallel} N_{\perp} \frac{mc^2}{T_{\parallel}} \sum_{n=1}^{+\infty} \frac{\lambda_{\perp}^{n-1}}{2^n n!} \sum_{k=0}^{+\infty} c_{n,k} \lambda_{\perp}^k \frac{(n+k+l)!}{(n+k)!l!} \begin{pmatrix} n \\ i(n+k) \end{pmatrix}$$
$$\times \frac{\partial}{\partial z_{\parallel}} \left[ \frac{T_{\perp}}{T_{\parallel}} W^{[\pm]}_{\frac{3}{2},n+k+l+1} + n\frac{\omega_c}{\omega} \Delta^{[\mp]}_{\frac{3}{2},n+k+i}(l, \frac{T_{\perp}}{T_{\parallel}}) \right]$$
$$-\frac{\bar{\omega}_p^2}{\bar{\omega}} N_{\parallel} N_{\perp} \frac{mc^2}{T_{\parallel}} \sum_{k=1}^{+\infty} c_{0,k} \lambda_{\perp}^{k-1} \frac{(k+l)!}{k!l!} \begin{pmatrix} 0 \\ -ik \end{pmatrix} \frac{\partial}{\partial z_{\parallel}} \left( \frac{T_{\perp}}{T_{\parallel}} W^{(0)}_{\frac{3}{2},k+l+1} \right) \quad (15)$$

$$\varepsilon_{zz} - 1 = -\left[\frac{\omega_p}{\omega}\right]^2 \frac{mc^2}{T_{\perp}} \sum_{n=1}^{+\infty} \frac{\lambda_{\perp}^n}{2^n n!} \sum_{k=0}^{+\infty} c_{n,k} \lambda_{\perp}^k \frac{(n+k+l)!}{(n+k)!l!}$$
$$\times \frac{\partial}{\partial N_{\parallel}} (N_{\parallel} [\frac{T_{\perp}}{T_{\parallel}} W^{(+)}_{\frac{3}{2},n+k+l+1} + n\frac{\omega_c}{\omega} \Delta^{(-)}_{\frac{3}{2},n+k+l}(l, \frac{T_{\perp}}{T_{\parallel}})])$$
$$-\left[\frac{\omega_p}{\omega}\right]^2 \frac{mc^2}{T_{\parallel}} \sum_{k=0}^{+\infty} c_{0,k} \lambda_{\perp}^k \frac{(k+l)!}{k!l!} \frac{\partial}{\partial N_{\parallel}} \left( N_{\parallel} W^{(0)}_{\frac{3}{2},k+l+1} \right) \quad (16)$$

The above expressions coincide with [4], i.e., in the weakly relativistic, $l$=0 thermal plasma case.

## References


[1] D. E. Baldwin, I. B. Berstein,M. P. M. Weenink, "*Kinetic Theory of Plasma Waves in a Magnetic Field*", Vol.**3**, Wiley, New York (1969).

[2] G. Bekefi," *Radiation Processes in Plasmas*", Weley, New York, (1966).

[3] S. Tamor,"Calculation of the Plasma dielectric Tensor for Arbitrary Distribution Functions",**EC6**, Oxford, (1986).

[4] M. Bornatici *et al.*, "Electron Cyclotron Emission and Absorption in Fusion Plasmas", Phys. Fluids,**23** $N_\circ$ 9, 1153(1983).

[5] I. P. Shkarofsky, "Dielectric Tensor in Vlasov Plasmas near Cyclotron Harmonics, Phys. Fluids, **9** $N_\circ$ 3,561 (1966).



*Aknowledgements*

This work was supported by: MPI, CNR, Italy.