A NEW REPRESENTATION OF THE RELATIVISTIC DIELECTRIC TENSOR
FOR A MAGNETIZED PLASMA

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A representation of the dielectric tensor which does not involve the
sum over harmonics is obtained. The novel form can be expressed in terms of
either a product of two Bessel functions whose orders are functions of
momentum, or an integral over the range (0,π/2) of combinations of J0,1,2
with cosine functions.

Introducing the standard form of the (relativistic) dielectric tensor
\[ \epsilon_{ij} \], relevant to electron cyclotron interaction in a magnetized plasma,
involves both a double integral over the momentum variables, e.g., \( \mathbf{p}_1 \) and
\( \mathbf{p}_0 \), and an infinite sum over harmonics of terms each of which contains the
product of two Bessel functions whose argument is a function of \( \mathbf{p}_1 \). More
specifically, in the reference frame in which \( \mathbf{B}_0 = \hat{z} B_0 \) and \( k \mathbf{k} = \hat{x} \mathbf{k}_x + \hat{y} \mathbf{k}_y + \hat{z} \mathbf{k}_z \), one has

\[ \epsilon_{ij} = \delta_{ij} + 2m c^2 \begin{vmatrix} \mathbf{E}_0 \end{vmatrix} \int \frac{d^3 \mathbf{p}_1}{(2\pi)^3} \int \frac{d^3 \mathbf{p}_0}{(2\pi)^3} \left[ \mathbf{F} (\mathbf{p}_1, \mathbf{p}_0) \mathbf{J}_2 (k c \mathbf{p}_0 \cdot \mathbf{z}) \right] \left[ \mathbf{u} (\mathbf{r}_0) \right] \mathbf{J}_1 (p_1) \mathbf{p}_1 \right] \]

(1)

where \( \mathbf{F}(\mathbf{p}_1, \mathbf{p}_0) = \mathbf{F}(\mathbf{p}_1) \cdot \mathbf{F}(\mathbf{p}_0) \) (propagator) and \( \mathbf{u}(\mathbf{r}_0) \) (prolate)
equilibrium distribution function, and \( \mathbf{J}_0 (\mathbf{p}_0 \cdot \mathbf{z}) \) (cyclotron).

The tensor \( \mathbf{\epsilon}_{ij} \) is given by an infinite sum over cyclotron harmonics of a
quantity which contains a singular factor multiplied by a dyadic tensor,
namely,

\[ \mathbf{\epsilon}_{ij} = \sum_n \mathbf{y}^{(n)} \mathbf{z}_n \mathbf{z}_n \]

(2)

with \( \mathbf{y}^{(n)} = \mathbf{y}^{(n)} \mathbf{z}_n \), the perpendicular (to \( \mathbf{B}_0 \)) component being

\[ \mathbf{\epsilon}_{ij} = \sum_n \mathbf{y}^{(n)} \mathbf{z}_n \mathbf{z}_n \]
the matrix appearing in (5) being (real and symmetric). The main features of the form (5) of the tensor $T_{ij}$ with respect to (2) are: i) the infinite sum over harmonics is no longer present (explicitly); ii) the orders of the Bessel functions appearing in (5) are now functions of the momentum via the same variable $\mu = (\omega \nu \sigma) (\gamma - \nu \Pi \Pi' / \nu \Pi' \sigma)$ that appears in the denominator $\sin(\mu \omega)$. Using (5) into the double integral over the momentum variables in (1) makes it convenient to choose $\mu$ as one of the integration variables, instead of $\Pi_1$ or $\Pi_{11}$, and split the range of the corresponding integral into symmetric intervals around each singular point $\mu = n \Pi$ (in integer), with the result that

$$
\int_{\mu=\infty}^{\infty} \frac{f(\mu) d\mu}{\sin(\mu \omega)} = \sum_{n=-\infty}^{\infty} \int_{\mu=\infty}^{\infty} \frac{f(\mu) d\mu}{\sin(\mu \omega)} = \sum_{n=-\infty}^{\infty} \left[ \int_{\mu=\infty}^{\infty} f(\mu) d\mu \right] \sin^{-1}(\mu / \omega)
$$

Since the matrix on the right-hand-side of (5) is real, the anti-Hermitian part of the dielectric tensor enters entirely from the singularities connected with $1 / \sin(\mu \omega)$. Thus, from (6) one recovers the standard harmonic expansion (3) for $F_{ij}$ and, hence, no real advantage is obtained by using (5) rather than (2). As for the Hermitian part of $T_{ij}$ instead, for every singular point $n$ the $\mu$-integration in (6) is over a finite range instead of being an infinite integral as in the standard harmonic expansion (2). This fact turns out to be advantageous for the numerical evaluation of $F_{ij}$ at high temperatures and high frequencies. (We note that the (2,3)-element of the (1,3)-element) obtained by Weiss and (Tamor) in the expression corresponding to (5) does not appear to be correct).

B. Integral representation of $T_{ij}$. Each element of matrix (5) can be expressed in integral form with a single Bessel function by using the relation

$$
J_\nu(z)J_{\nu}'(z) = \frac{1}{\pi} \int_0^{\pi} J_{\nu+\lambda}(2z \cos \psi) \cos[(\nu-\lambda) \psi], \quad Re(\nu+\lambda) > 0
$$

Applying (7) to (5) yields the symmetric form
\[ T_{ij} = \frac{2^{n/2}}{\sin(n\phi)} \left[ \frac{\cos(2n\Phi)}{2^{n/2}} \right] \left[ \begin{array}{c} \frac{1}{2} J_0(2\Phi) \\ \frac{1}{2} J_2(2\Phi) \\ \vdots \\ \frac{1}{2} J_{2n}(2\Phi) \end{array} \right] - \frac{\cos((2n+1)\Phi)}{2^{n/2}} \left[ \begin{array}{c} \frac{1}{2} J_0((2n+1)\Phi) \\ \frac{1}{2} J_2((2n+1)\Phi) \\ \vdots \\ \frac{1}{2} J_{2n+2}((2n+1)\Phi) \end{array} \right] \]

with \( t_{23} = \frac{p_0}{p_1} \left( \frac{\sqrt{2}}{1 - \cos(2\Phi)} \right)^{1/2} \), the argument of \( J_{0,1,2} \) being \( 2\nu \cos \Phi = 2(\omega/\omega_c)\nu \). A representation similar to (8) has been obtained also by Shi et al.\(^6\). The salient characteristic of (8) is the \( \Phi \)-integral over the \((0,\pi)\)-range of combinations of \( J_{0,1,2} \) with cosine functions. As for (5) the poles connected with \( \sin(n\phi)\neq 0 \) contribute an anti-Hermitian part to the dielectric tensor. More specifically, for \( n^2 \geq 1 \) one has

\[ \left[ \begin{array}{c} \nu \left( f_{\alpha} \right) \end{array} \right]_{n} = \left[ \begin{array}{c} \frac{p_0}{p_1} \left( \frac{\sqrt{2}}{1 - \cos(2\Phi)} \right)^{1/2} \end{array} \right]_{n} \left[ \begin{array}{c} \nu \left( f_{\alpha} \right) \end{array} \right]_{n} \]

where \( t_{ij} \) is the integrand of the \( \Phi \)-integration in (8).

Acknowledgments. This work has been performed with financial support from the Ministero della Pubblica Istruzione of Italy.

References