europhysics conference abstracts

15th European Conference on

Controlled Fusion and Plasma Heating

Dubrovnik, May 16—20, 1988

Editors: S. Pešić, J. Jacquinot

Contributed Papers
Part III

Published by: European Physical Society

Series Editor: Dr. J. Heijn, Petten

Managing Editor: G. Thomas, Geneva

VOLUME
12B
Part III

A NEW REPRESENTATION OF THE RELATIVISTIC DIELECTRIC TENSOR FOR A MAGNETIZED PLASMA

M. Bornatici and G. Chiozzi
Physics Department, University of Ferrara, Ferrara, Italy

A representation of the dielectric tensor which does not involve the sum over harmonics is obtained. The novel form can be expressed in terms of either a product of two Bessel functions whose orders are functions of momentum, or an integral over the range $(0,\pi/2)$ of combinations of $J_{0,1,2}$ with cosine functions.

Introduction. The standard form of the (relativistic) dielectric tensor ϵ_{ij}, relevant to electron cyclotron interaction in a magnetized plasma, involves both a double integral over the momentum variables, e.g., $p_\perp$ and $p_\parallel$, and an infinite sum over harmonics of terms each of which contains the product of two Bessel functions whose argument is a function of $p_\perp$. More specifically, in the reference frame in which $\underline{B}_o = \hat{z}B_o$ and $\underline{k}=k_\perp\hat{x}+k_\parallel\hat{z}$, one has[1]

$$\epsilon_{ij} = \delta_{ij}+2\pi\left[\frac{\omega_p}{\omega}\right]^2\int_0^\infty dp_\perp\int_{-\infty}^{\infty} dp_\parallel\left[\frac{p_\parallel}{\gamma} F(p_\perp,p_\parallel)\delta_{iz}\delta_{jz}+\frac{\omega}{\omega_c} U(f_o)T_{ij}(p_\perp,p_\parallel)\right] \qquad (1)$$

where $F(p_\perp,p_\parallel)\equiv p_\perp(\partial f_o/\partial p_\parallel)-p_\parallel(\partial f_o/\partial p_\perp)$, with $f_o=f_o(p_\perp,p_\parallel)$ the (gyrotropic) equilibrium distribution ($\int d^3p\ f_o=1$), and $U(f_o)\equiv\partial f_o/\partial p_\perp+(N_\parallel/\gamma mc)F(p_\perp,p_\parallel)$. The tensor T_{ij} is given by an infinite sum over cyclotron harmonics of a quantity which contains a singular factor multiplying a dyadic tensor, namely,

$$T_{ij}=\sum_n \frac{V_i^{(n)}(V_j^{(n)})^*}{\mu-n}, \quad \mu \equiv \frac{\omega}{\omega_c}(\gamma-N_\parallel p_\parallel/mc) \qquad (2)$$

with $\underline{V}^{(n)}=\underline{V}_\perp^{(n)}+\hat{z}p_\parallel J_n$, the perpendicular (to $\underline{B}_o$) component being

$v_\perp^{(n)} = p_\perp[\hat{x}(nJ_n/b)+i\hat{y}J_n'] = (p_\perp/2)[(\hat{x}+i\hat{y})J_{n-1}+(\hat{x}-i\hat{y})J_{n+1}]$, the later representation referring to a frame of reference in which the wave electric field perpendicular to $\underline{B}_o$ is $\underline{E}_\perp = (1/\sqrt{2})(E_x+iE_y, E_x-iE_y)$. In this reference frame, to be referred to as the "circular frame", the dyadic in (2) is real. Furthermore, $J_n \equiv J_n(b)$, $J_n' \equiv dJ_n/db$, with $b \equiv (\omega/\omega_c)N_\perp(p_\perp/mc)$.

The infinite number of singularities in p-space arising from the factor $(\mu-n)^{-1}$ in (2) and connected with the relativistic resonance condition $\gamma - N_\parallel p_\parallel/mc - n\omega_c/\omega = 0$ is dealt with by performing the integration along the causal contour which amounts to assuming that $N_\parallel$ has a small (negative) imaginary part (ω is taken real) so that, using the Plemelj formula, $[\mu(\gamma, p_\parallel)-n]^{-1} = P(1/(\mu-n)) - i\pi\delta(\mu-n)$, P indicating the Cauchy principal value. It follows that one can write $T_{ij} = T_{h,ij} + iT_{a,ij}$, respectively the Hermitian and anti-Hermitian part, so that, in particular, the anti-Hermitian part of the dielectric tensor (1) is

$$\epsilon_{a,ij} = -2\pi^2 \left[\frac{\omega_p}{\omega}\right]^2 \int dp_\perp dp_\parallel U(f_o) \sum_n v_i^{(n)} (v_j^{(n)})^* \delta(\gamma - N_\parallel p_\parallel/mc - n\omega_c/\omega) \qquad (3)$$

whose salient characteristic is the infinite sum over cyclotron harmonics.

Representation without expansion in harmonics. The tensor T_{ij} and, hence, the dielectric tensor ϵ_{ij} can be expressed in forms whose Hermitian parts do not involve the (infinite) sum over harmonics, thus being possibly more advantageous than (2).

A. Two Bessel functions representation of T_{ij}. The summation of the harmonic series appearing in (2) can be performed by means of the summation rule[2]

$$\sum_{n=-\infty}^{\infty} \frac{J_{n+p}(b)J_{n+1}(b)}{\mu-n} = \frac{\pi}{\sin[\pi(\mu+1)]} J_{\mu+p}(b)J_{-(\mu+1)}(b), \quad \mathrm{Re}(p-1) \geq -1 \qquad (4)$$

With (4), the tensor (2) reduces to

$$T_{ij} = -\frac{\pi p_\perp}{\sin(\pi\mu)} \begin{bmatrix} \frac{p_\perp}{2} J_{\mu+1}J_{-(\mu+1)} & \frac{p_\perp}{2} J_{\mu+1}J_{1-\mu} & -\frac{p_\parallel}{\sqrt{2}} J_{-\mu}J_{\mu+1} \\ \frac{p_\perp}{2} J_{\mu+1}J_{1-\mu} & \frac{p_\perp}{2} J_{\mu-1}J_{1-\mu} & -\frac{p_\parallel}{\sqrt{2}} J_{-\mu}J_{\mu-1} \\ -\frac{p_\parallel}{\sqrt{2}} J_{-\mu}J_{\mu+1} & -\frac{p_\parallel}{\sqrt{2}} J_{-\mu}J_{\mu-1} & -\frac{p_\parallel^2}{p_\perp} J_\mu J_{-\mu} \end{bmatrix} \qquad (5)$$

the matrix appearing in (5) being (real and) symmetric. The main features of the form (5) of the tensor T_{ij} with respect to (2) are: i) the infinite sum over harmonics is no longer present (explicitly); ii) the orders of the Bessel functions appearing in (5) are now functions of the momentum via the same variable μ ($\equiv(\omega/\omega_c)(\gamma - N_\parallel p_\parallel/mc)$) that appears in the denominator $\sin(\pi\mu)$. Using (5) into the double integral over the momentum variables in (1) makes it convenient to chose μ as one of the integration variables, instead of $p_\perp$ or $p_\parallel$, and split the range of the corresponding integral into symmetric intervals around each singular point $\mu=n$ (n integer), with the result that

$$\int_{-\infty}^{\infty} d\mu \frac{f(\mu)}{\sin(\pi\mu)} = \sum_{n=-\infty}^{\infty} \int_{n-1/2}^{n+1/2} d\mu \frac{f(\mu)}{\sin(\pi\mu)} = \sum_n \left[P\int_{n-1/2}^{n+1/2} d\mu \frac{f(\mu)}{\sin(\pi\mu)} - i(-1)^n f(n) \right] \qquad (6)$$

Since the matrix on the right-hand-side of (5) is real, the anti-Hermitian part of the dielectric tensor comes entirely from the singularities connected with $1/\sin(\pi\mu)$. Thus, from (6) one recovers the standard harmonic expansion (3) for $\epsilon_{a,ij}$ and, hence, no real advantage is obtained by using (5) rather than (2). As for the Hermitian part of ϵ_{ij} instead, for every singular point n the μ-integration in (6) is over a finite range instead of being an infinite integral as in the standard harmonic expansion (2). This fact turns out to be advantageous for the numerical evaluation of $\epsilon_{h,ij}$ at high temperatures and high frequencies[3,4] (We note that the (2,3)-element (the (1,3)-element) obtained by Weiss[3] (Tamor[4]) in the expression corresponding to (5) does not appear to be correct).

B. Integral representation of T_{ij}. Each element of matrix (5) can be expressed in integral form with a single Bessel function by using the relation[5]

$$J_\nu(z)J_\lambda(z) = \frac{2}{\pi}\int_0^{\pi/2} d\phi\, J_{\nu+\lambda}(2z\cos\phi)\cos[(\nu-\lambda)\phi], \quad \mathrm{Re}(\nu+\lambda) > -1 \qquad (7)$$

Applying (7) to (5) yields the symmetric form

$$T_{ij} = -\frac{2p_\perp}{\sin(\pi\mu)} \int_0^{\pi/2} d\phi \begin{bmatrix} \frac{p_\perp}{2} J_0 \cos[2(\mu+1)\phi] & \frac{p_\perp}{2} J_2 \cos[2\mu\phi] & -\frac{p_\parallel}{\sqrt{2}} J_1 \cos[(2\mu+1)\phi] \\ \frac{p_\perp}{2} J_2 \cos(2\mu\phi) & \frac{p_\perp}{2} J_0 \cos[2(\mu-1)\phi] & t_{23} \\ -\frac{p_\parallel}{\sqrt{2}} J_1 \cos[(2\mu+1)\phi] & t_{23} & -\frac{p_\parallel^2}{p_\perp} J_0 \cos[2\mu\phi] \end{bmatrix} \quad (8)$$

with $t_{23} = -(p_\parallel/\sqrt{2})\{(2\mu/b)J_0\cos(2\mu\phi) - J_1\cos[(2\mu+1)\phi]\}$; the argument of $J_{0,1,2}$ being $2b\cos\phi = 2(\omega/\omega_c)N_\perp(p_\perp/mc)\cos\phi$ (A representation similar to (8) has been obtained also by Shi et al.[6]). The salient characteristic of (8) is the ϕ-integral over the $(0,\pi/2)$-range of combinations of $J_{0,1,2}$ with cosine functions. As for (5) the poles connected with $\sin(\pi\mu)=0$ contribute an anti-Hermitian part to the dielectric tensor. More specifically, for $N_\parallel^2<1$ one has

$$\epsilon_{a,ij} = 4\pi \left[\frac{\omega_p}{\omega}\right]^2 \sum_n (-1)^n \int_0^{\pi/2} d\phi \int_0^{p_o} dp_\perp \frac{p_\perp}{\Delta} \left[\gamma U(f_o) t_{ij}\right]_{\mu=n} \quad (9)$$

where t_{ij} is the integrand of the ϕ-integration in (8), $\Delta = [(1-N_\parallel^2)(p_o^2 - p_\perp^2)]^{1/2}/(mc)^2$ and $p_o = [(n\omega_c/\omega)^2/(1-N_\parallel^2) - 1]^{1/2} mc$.

Acknowledgements. This work has been performed with financial support from the Ministero della Pubblica Istruzione of Italy.

References

1. M.Bornatici, R.Cano, O.De Barbieri, F.Engelmann, Nucl.Fusion 23, 1153(1983)
2. E.A.Evangelidis, J.Math.Phys. 25, 2151(1984)
3. I.Weiss, J.Comp.Phys. 61, 403(1985)
4. S.Tamor, Int.Rep. SAIC-86/3081-APPAT-77
5. I.S.Gradshteyn and I.A.Ryzhik, Table of Integrals, Series and Products (Academic Press, New York, 1980)
6. B.R.Shi, J.D.Gaffey, Jr. and C.S.Wu, Phys. Fluids 29, 4212(1986)