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Weakly relativistic dielectric tensor in the presence of temperature anisotropy

By M. BORNATICI, G. CHIOZZI and P. DE CHIARA
Dipartimento di Fisica dell'Università di Ferrara, 44100 Ferrara, Italy

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Analytical expressions for the weakly relativistic dielectric tensor near the electron-cyclotron frequency and harmonics are obtained to any order in finite-Larmor-radius effects for a bi-Maxwellian distribution function. The dielectric tensor is written in terms of generalized Shkarofsky dispersion functions, whose properties are well known. Relevant limiting cases are considered and, in particular, the anti-Hermitian part of the (fully relativistic) dielectric tensor is evaluated for two cases of strong temperature anisotropy.

1. Introduction

One of the standard forms of the (relativistic) dielectric tensor ϵ_{ij}, relevant to the study of the interaction between magnetized plasmas and electromagnetic waves, is characterized by integration over the momentum variables, e.g. $p_\perp$ and $p_\parallel$, the momenta perpendicular and parallel to the magnetic field, a time integration and an infinite sum over harmonics of terms containing the product of two Bessel functions whose argument depends on $p_\perp$ (Bornatici *et al.* 1983*a*). More specifically, in the reference frame in which the equilibrium magnetic field $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$ and the wave vector $\mathbf{k} = k_\perp\hat{\mathbf{x}} + k_\parallel\hat{\mathbf{z}}$, the relativistic dielectric tensor ϵ_{ij} can be written in the form (only the contribution from electrons is considered)

$$\epsilon_{ij} = \delta_{ij} - 2\pi i\left(\frac{\omega_p}{\omega}\right)^2 \sum_{n=-\infty}^{\infty}\int_0^\infty dt \int dp_\perp\, dp_\parallel\, p_\perp^2 \exp\left[it\left(\gamma - N_\parallel\frac{p_\parallel}{mc} - \frac{n\omega_c}{\omega}\right)\right]$$
$$\times V_i^{(n)}(V_j^{(n)})^* \hat{U} f_0 + 2\pi\left(\frac{\omega_p}{\omega}\right)^2 \delta_{iz}\sum_{n=-\infty}^{\infty}\int dp_\perp\, dp_\parallel \frac{p_\perp}{\gamma} V_j^{(n)} J_n \hat{F} f_0, \quad (1)$$

where
$$\mathbf{V}^{(n)} \equiv \left(\frac{n}{b}J_n(b),\, iJ_n'(b),\, \frac{p_\parallel}{p_\perp}J_n(b)\right), \quad (2)$$

$$\hat{U} \equiv \frac{\partial}{\partial p_\perp} + \frac{N_\parallel}{\gamma mc}\hat{F}, \quad \hat{F} \equiv p_\perp\frac{\partial}{\partial p_\parallel} - p_\parallel\frac{\partial}{\partial p_\perp}, \quad (3)$$

with $J_n(b)$ the Bessel function of the first kind of order n (the harmonic number) and argument $b \equiv N_\perp(\omega/\omega_c)p_\perp/mc$, and $J_n'(b) \equiv dJ_n(b)/db$; $(\mathbf{V}^{(n)})^*$ denotes the complex conjugate of $\mathbf{V}^{(n)}$. Furthermore, $\gamma \equiv [1 + (p/mc)^2]^{\frac{1}{2}}$ and $f_0 \equiv f_0(p_\perp, p_\parallel)$ is the (gyrotropic) equilibrium distribution function of the electrons, normalized to unity; $N_\perp$ and $N_\parallel$ are the wave refractive indices respectively perpendicular and parallel to the magnetic field, and ω_p and ω_c (>0) are respectively the plasma and cyclotron frequencies of the electrons.

The last term on the right-hand side of (1) requires some comment. Of course, it is identically zero for an isotropic distribution function, i.e. for $\hat{F}f_0 = 0$. When summed over harmonics, the corresponding off-diagonal elements are identically zero and only the (z, z) element survives; in fact, explicitly,

$$\sum_{n=-\infty}^{\infty} V_j^{(n)} J_n = \frac{p_\parallel}{p_\perp} \delta_{jz}. \tag{4}$$

It is only when considered along with (4), but not term by term, that the dielectric tensor (1) possesses the symmetries relations $\epsilon_{zx} = \epsilon_{xz}$ and $\epsilon_{zy} = -\epsilon_{yz}$. In this paper the form (1) together with (4) is used. On the other hand, the anti-Hermitian part of the dielectric tensor, $\epsilon_{a,ij}$ is connected solely with the second term on the right-hand side of (1). More specifically, integration over t (Bekefi 1966) produces the resonant denominator $\gamma - N_\parallel (p_\parallel/mc) - (n\omega_c/\omega)$, and in the limit in which the imaginary part of ω and/or k is small (so that the Plemelj formula is applicable), one finds

$$\epsilon_{a,ij} = -2\pi^2 \left(\frac{\omega_p}{\omega}\right)^2 \int dp_\perp \, dp_\parallel \, p_\perp^2 (\hat{U} f_0) \sum_{n=-\infty}^{\infty} V_i^{(n)} (V_j^{(n)})^* \, \delta\!\left(\gamma - \frac{N_\parallel p_\parallel}{mc} - \frac{n\omega_c}{\omega}\right). \tag{5}$$

From (3) and (5) it appears that the effect of the anisotropy of the distribution function, accounted for through $\hat{F}f_0$, enters in combination with $N_\parallel$.

The possibility of analytically performing the integration over momentum variables as well as the sum over the harmonic numbers in (1) and (5) depends on the actual form of the distribution function f_0. In particular, for a (relativistic) loss-cone-type distribution, a fully relativistic calculation can be performed analytically, with the sole exception of the t integration in (1) (cf. for example the second of Trubnikov's formulae for a relativistic Maxwellian (Bornatici *et al.* 1983*a*)); in particular, in the weakly relativistic limit ($\gamma \approx 1 + p^2/2m^2c^2, mc^2/T \gg 1$), ϵ_{ij} can be expressed in terms of the generalized Shkarofsky functions studied by Robinson (1987). The anti-Hermitian part (5) of the dielectric tensor can also be expressed in the form of an infinite sum over harmonics of terms, each of which contains modified Bessel functions (Bornatici & Ruffina 1985; Ziebell 1988).

For a bi-Maxwellian ($T_\parallel \neq T_\perp$) distribution, on the other hand, only one of the integrations in (1) can be carried out analytically in the fully relativistic case (Tsang 1984), whereas in the weakly relativistic limit both momentum integrations can again be performed analytically (Tsai *et al.* 1981; Lam, Scharer & Audenaerde 1984), as shown in §2. Relevant limiting cases of the general dielectric tensor thus obtained are considered in §3, while the limits of strong temperature anisotropy are investigated in §4. In particular, with reference to $\epsilon_{a,ij}$ only, the integration over momentum space can be carried out analytically, for the fully relativistic case, in the limit of strong temperature anisotropy, i.e. for $T_\perp \neq 0$ and $T_\parallel = 0$ (two-dimensional Maxwellian), or $T_\parallel \neq 0$ and $T_\perp = 0$ (one-dimensional Maxwellian). These cases are also considered in §4. Our conclusions are summarized in §5. A few mathematical details of the calculation are given in Appendices A and B.

2. Weakly relativistic dielectric tensor for a bi-Maxwellian distribution

In the weakly relativistic approximation and for the bi-Maxwellian distribution function

$$f_0(p_\perp, p_\|) = \frac{1}{(2\pi m)^{\frac{3}{2}}} \frac{1}{T_\|^{\frac{1}{2}} T_\perp} \exp\left(-\frac{p_\|^2}{2mT_\|} - \frac{p_\perp^2}{2mT_\perp}\right) \tag{6}$$

($T_\|$ and $T_\perp$ being the temperatures respectively parallel and perpendicular to the magnetic field) both the $p_\perp$ and $p_\|$ integrations in (1) can be carried out analytically, as shown in detail in Appendix A, and the dielectric tensor can be expressed in terms of the dispersion function

$$R_{q,l}^{(n)}\left(z_n(T_\|), a_\|, \lambda_\perp; \frac{T_\perp}{T_\|}\right) \equiv -i\int_0^\infty d\tau \frac{\exp\left[iz_n(T_\|)\tau - a_\| \tau^2/(1-i\tau)\right]}{(1-i\tau)^q (1-i\tau T_\perp/T_\|)^l} \Gamma_n\left(\frac{\lambda_\perp}{1-i\tau T_\perp/T_\|}\right), \tag{7}$$

with q and l respectively half-integer and integer positive numbers,

$$\left.\begin{aligned} z_n(T_\|) &\equiv \mu_\|\left(1-\frac{n\omega_c}{\omega}\right), \quad a_\| \equiv \tfrac{1}{2}\mu_\| N_\|^2, \quad \mu_{\|(\perp)} \equiv \frac{mc^2}{T_{\|(\perp)}} \quad (\gg 1), \\ \Gamma_n(x) &\equiv e^{-x} I_n(x), \quad \lambda_\perp \equiv \left(\frac{\omega}{\omega_c}\right)^2 \frac{N_\perp^2}{\mu_\perp}, \end{aligned}\right\} \tag{8}$$

$I_n(x)$ being a modified Bessel function. With reference to (7), one finds the following.

(i) The effect of the temperature anisotropy is present explicitly both in the denominator of the integrand and in the argument of the Γ_n functions.

(ii) In the isotropic limit, i.e. $T_\| = T_\perp$, (7) reduces to the generalized Shkarofsky function studied by Robinson (1987).

(iii) In the non-relativistic limit, i.e. $(1-i\tau) \approx (1-i\tau T_\perp/T_\|) \approx 1$,

$$R_{q,l}^{(n)} \to -\frac{1}{2a_\|^{\frac{1}{2}}} Z\left(\frac{z_n(T_\|)}{2a_\|^{\frac{1}{2}}}\right)\Gamma_n(\lambda_\perp),$$

where Z is the usual non-relativistic dispersion function.

(iv) Using the series representation of $\Gamma_n(x)$ given by (A 6), one can express $R_{q,l}^{(n)}$ in terms of a series in the Larmor parameter $\lambda_\perp$, i.e.

$$R_{q,l}^{(n)} = \sum_{k=0}^{\infty} a_{k,|n|} \lambda_\perp^{k+|n|} W_{q,k+l+|n|}, \tag{9}$$

where

$$a_{k,|n|} = \frac{(-1)^k [2(|n|+k)]!}{(|n|+k)!\,(2|n|+k)!\,k!} 2^{-(k+|n|)},$$

and

$$W_{q,p}\left(z_n(T_\|), a_\|; \frac{T_\perp}{T_\|}\right) \equiv -i\int_0^\infty d\tau \frac{\exp\left[iz_n(T_\|)\tau - a_\| \tau^2/(1-i\tau)\right]}{(1-i\tau)^q (1-i\tau T_\perp/T_\|)^p} \tag{10a}$$

is a generalization to the anisotropic-temperature case of the weakly relativistic Shkarofsky dispersion function (Shkarofsky 1966)

$$W_q(z_n(T_\|), a_\|) \equiv -i\int_0^\infty d\tau \frac{\exp\left[iz_n(T_\|)\tau - a_\| \tau^2/(1-i\tau)\right]}{(1-i\tau)^q}, \tag{10b}$$

whose analytical properties are well known (Maroli & Petrillo 1981; Krivenski & Orefice 1983; Shkarofsky 1986; Robinson 1986).

(v) Considering explicitly the case for which $|1-T_\parallel/T_\perp| \leqslant 1$ and expressing $(1-i\tau T_\perp/T_\parallel)^{-p}$ in terms of the corresponding series representation, i.e.

$$\left(1-\frac{i\tau T_\perp}{T_\parallel}\right)^{-p} = \left(\frac{T_\parallel}{T_\perp}\right)^p \sum_{k=0}^{\infty} \frac{(p+k-1)!}{k!(p-1)!}\left(1-\frac{T_\parallel}{T_\perp}\right)^k (1-i\tau)^{-(p+k)},$$

one has, from (10*a*),

$$W_{q,p} = \left(\frac{T_\parallel}{T_\perp}\right)^p \sum_{k=0}^{\infty} \frac{(p+k-1)!}{k!(p-1)!}\left(1-\frac{T_\parallel}{T_\perp}\right)^k W_{q+p+k}. \tag{11}$$

(vi) From (11) and the expression for the imaginary part of W_q (Maroli & Petrillo 1981), one has for the imaginary part of $W_{q,p}$,

$$\operatorname{Im} W_{q,p} = -\pi\left(\frac{T_\parallel}{T_\perp}\right)^p \exp\{-[\mu_\parallel N_\parallel^2 - z_n(T_\parallel)]\} \sum_{k=0}^{\infty} \frac{(p+k-1)!}{k!(p-1)!}\left(1-\frac{T_\parallel}{T_\perp}\right)^k \times \left(\frac{x_n}{\mu_\parallel N_\parallel^2}\right)^{q+p+k-1} I_{q+p+k-1}(x_n), \tag{12}$$

with

$$x_n \equiv \mu_\parallel N_\parallel \left[N_\parallel^2 + 2\left(\frac{n\omega_c}{\omega}-1\right)\right]^{\frac{1}{2}}$$

and

$$N_\parallel^2 + 2\left(\frac{n\omega_c}{\omega}-1\right) > 0, \tag{13}$$

whereas $\operatorname{Im} W_{q,p} = 0 \quad \text{for } N_\parallel^2 + 2(n\omega_c/\omega - 1) \leqslant 0.$

In particular, in the non-relativistic limit, i.e. for

$$N_\parallel^2 \gg \operatorname{Max}\left\{2\left|\frac{n\omega_c}{\omega}-1\right|, \frac{1}{\mu_\parallel}\right\}, \tag{14}$$

by using the asymptotic expansion of the I_q function in (12), one obtains

$$\operatorname{Im} W_{q,p} = -\left(\frac{\pi}{2\mu_\parallel N_\parallel^2}\right)^{\frac{1}{2}} \exp\left[-\frac{\mu_\parallel}{2N_\parallel^2}\left(\frac{n\omega_c}{\omega}-1\right)^2\right] \equiv -\left(\frac{1}{2\mu_\parallel N_\parallel^2}\right)^{\frac{1}{2}} \operatorname{Im} Z(\zeta_n), \tag{15}$$

where $Z(\zeta_n)$ is the familiar non-relativistic plasma dispersion function of argument $\zeta_n \equiv (\frac{1}{2}\mu_\parallel)^{\frac{1}{2}}(\omega-n\omega_c)/k_\parallel c$. Note that (15) is independent of the perpendicular temperature.

For the explicit expression of the dielectric tensor for the distribution (6), one finds in terms of the functions (7), and to arbitrary order in finite-Larmor-radius (FLR) effects,

$$\begin{Bmatrix} \epsilon_{xx}-1 \\ \epsilon_{xy} = -\epsilon_{yx} \\ \epsilon_{yy}-1 \end{Bmatrix} = -\left(\frac{\omega_p}{\omega}\right)^2 \frac{mc^2}{T_\parallel} \sum_n$$

$$\left[\begin{Bmatrix} \dfrac{n^2}{\lambda_\perp} R^{(n)}_{\frac{1}{2},1} \\ -inR'^{(n)}_{\frac{1}{2},2} \\ \dfrac{n^2}{\lambda_\perp} R^{(n)}_{\frac{1}{2},1} - 2\lambda_\perp R'^{(n)}_{\frac{1}{2},3} \end{Bmatrix} + N_\parallel^2\left(1-\frac{T_\perp}{T_\parallel}\right)\frac{\partial}{\partial z_\parallel} \begin{Bmatrix} \dfrac{n^2}{\lambda_\perp} R^{(n)}_{\frac{3}{2},1} \\ -inR'^{(n)}_{\frac{3}{2},2} \\ \dfrac{n^2}{\lambda_\perp} R^{(n)}_{\frac{3}{2},1} - 2\lambda_\perp R'^{(n)}_{\frac{3}{2},3} \end{Bmatrix}\right., \tag{16a}$$

$$\begin{Bmatrix} \epsilon_{xz} = \epsilon_{zx} \\ \epsilon_{yz} = -\epsilon_{zy} \end{Bmatrix} = \left(\frac{\omega_p}{\omega}\right)^2 |N_\parallel| \left(\frac{T_\perp}{T_\parallel}\right)^{\frac{1}{2}} \left(\frac{mc^2}{T_\parallel}\right)^{\frac{3}{2}} \sum_n \left[1 - \frac{n\omega_c}{\omega}\left(1 - \frac{T_\parallel}{T_\perp}\right)\right] \frac{\partial}{\partial z_\parallel} \begin{Bmatrix} \frac{n}{\lambda_\perp^{\frac{1}{2}}} R^{(n)}_{\frac{3}{2},1} \\ i\lambda_\perp^{\frac{1}{2}} R'^{(n)}_{\frac{3}{2},2} \end{Bmatrix}, \quad (16b)$$

$$\epsilon_{zz} = 1 - \left(\frac{\omega_p}{\omega}\right)^2 \frac{mc^2}{T_\parallel} \sum_n \left[1 - \frac{n\omega_c}{\omega}\left(1 - \frac{T_\parallel}{T_\perp}\right)\right] \frac{\partial}{\partial N_\parallel} (N_\parallel R^{(n)}_{\frac{3}{2},1}), \quad (16c)$$

where $z_\parallel \equiv z_n(T_\parallel)$ and $R'^{(n)}_{q,l}$ is as in (7), but with $\Gamma'_n(x) \equiv d\Gamma_n(x)/dx$ in place of $\Gamma_n(x)$. In particular, using (A 7), one can express $R'^{(n)}_{q,l}$ in terms of a series, in analogy to (9):

$$R'^{(n)}_{q,l} = \sum_{k=0}^{\infty} (k+|n|)\, a_{k,|n|}\, \lambda_\perp^{k+|n|-1}\, W_{q,k+l+|n|-1}. \quad (17)$$

Note that from (7) it follows that

$$\frac{\partial R^{(n)}_{q,l}}{\partial z_\parallel} = R^{(n)}_{q,l} - R^{(n)}_{q-1,l},$$

$$\frac{\partial R^{(n)}_{q,l}}{\partial N_\parallel} = \mu_\parallel N_\parallel (R^{(n)}_{q-1,l} - 2R^{(n)}_{q,l} + R^{(n)}_{q+1,l}),$$

with $q \geq \frac{3}{2}$. From (16) it appears that the temperature-anisotropy effect connected with $1 - T_\perp/T_\parallel$ enters combination with either $N_\parallel$ or $n\omega_c/\omega$.

A convenient form of the dielectric tensor (16) is the *series representation*, which is obtained by applying (9) and (17):

$$\begin{Bmatrix} \epsilon_{xx} - 1 \\ \epsilon_{xy} = -\epsilon_{yx} \end{Bmatrix} = -\left(\frac{\omega_p}{\omega}\right)^2 \frac{mc^2}{T_\parallel} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \begin{Bmatrix} n^2 \\ -in(k+n) \end{Bmatrix} a_{k,n} \lambda_\perp^{k+n-1}$$
$$\times \left[W^{(\pm)}_{\frac{1}{2},k+n+1} + N_\parallel^2 \left(1 - \frac{T_\perp}{T_\parallel}\right) \frac{\partial}{\partial z_\parallel} W^{(\pm)}_{\frac{3}{2},k+n+1} \right], \quad (18a)$$

$$\epsilon_{yy} = 1 - \left(\frac{\omega_p}{\omega}\right)^2 \frac{mc^2}{T_\parallel} \left\{ \sum_{k=2}^{\infty} \frac{2(k-1)\,k^2}{2k-1} a_{k,0} \lambda_\perp^{k-1} \right.$$
$$\times \left[W_{\frac{1}{2},k+1} + N_\parallel^2 \left(1 - \frac{T_\perp}{T_\parallel}\right) \frac{\partial}{\partial z_\parallel} W_{\frac{3}{2},k+1} \right]_{n=0} + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \left[(k+n)^2 \frac{k(k+2n)}{2k+2n-1} \right]$$
$$\left. \times\, a_{k,n} \lambda_\perp^{k+n-1} \left[W^{(+)}_{\frac{1}{2},k+n+1} + N_\parallel^2 \left(1 - \frac{T_\perp}{T_\parallel}\right) \frac{\partial}{\partial z_\parallel} W^{(-)}_{\frac{3}{2},k+n+1} \right] \right\}, \quad (18b)$$

$$\begin{Bmatrix} \epsilon_{xz} = \epsilon_{zx} \\ \epsilon_{yz} = -\epsilon_{zy} \end{Bmatrix} = \left(\frac{\omega_p}{\omega}\right)^2 \frac{\omega}{\omega_c} |N_\parallel| N_\perp \frac{T_\perp}{T_\parallel} \frac{mc^2}{T_\parallel} \left[\begin{Bmatrix} 0 \\ i \sum_{k=1}^{\infty} k a_{k,0} \lambda_\perp^{k-1} \left[\frac{\partial}{\partial z_\parallel} W_{\frac{3}{2}}, k+1 \right]_{n=0} \end{Bmatrix} \right.$$
$$\left. + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \begin{Bmatrix} n \\ i(k+n) \end{Bmatrix} a_{k,n} \lambda_\perp^{k+n-1} \left[\frac{\partial}{\partial z_\parallel} W^{(\pm)}_{\frac{3}{2},k+n+1} - \frac{n\omega_c}{\omega} \left(1 - \frac{T_\parallel}{T_\perp}\right) \frac{\partial}{\partial z_\parallel} W^{(\mp)}_{\frac{3}{2},k+n+1} \right] \right], \quad (18c)$$

$$\epsilon_{zz} = 1 - \left(\frac{\omega_p}{\omega}\right)^2 \frac{mc^2}{T_\parallel} \left\{ \sum_{k=0}^{\infty} a_{k,0} \lambda_\perp^{k} \left[\frac{\partial}{\partial N_\parallel} (N_\parallel W_{\frac{3}{2},k+1}) \right]_{n=0} + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} a_{k,n} \lambda_\perp^{k+n} \right.$$
$$\left. \times \left[\frac{\partial}{\partial N_\parallel} (N_\parallel W^{(+)}_{\frac{3}{2},k+n+1}) - \frac{n\omega_c}{\omega} \left(1 - \frac{T_\parallel}{T_\perp}\right) \frac{\partial}{\partial N_\parallel} (N_\parallel W^{(-)}_{\frac{3}{2},k+n+1}) \right] \right\}, \quad (18d)$$

where $$W^{(\pm)}_{q,p} \equiv W_{q,p}\left(z_n(T_\parallel), a_\parallel; \frac{T_\perp}{T_\parallel}\right) \pm W_{q,p}\left(z_{-n}(T_\parallel), a_\parallel; \frac{T_\perp}{T_\parallel}\right), \tag{19}$$

with $W_{q,p}$ defined by (10a). The upper (lower) signs in (18a, c) refer to the upper (lower) term on the left-hand side of the corresponding equation. In the isotropic limit the dielectric tensor (18a–d) reduces to the expression obtained by Shkarofsky (1986).

It is worth noting that, once the dielectric tensor for a bi-Maxwellian distribution is known, one can obtain the corresponding dielectric tensor for the loss-cone distribution function (including temperature anisotropy)

$$f_0(p_\perp, p_\parallel) = \frac{1}{\pi^{\frac{3}{2}} l!\,(2mT_\parallel)^{\frac{1}{2}}\,2mT_\perp}\left(\frac{p_\perp^2}{2mT_\perp}\right)^l \exp\left(-\frac{p_\parallel^2}{2mT_\parallel} - \frac{p_\perp^2}{2mT_\perp}\right) \tag{20a}$$

through the relation (Tsai *et al.* 1981; Lam *et al.* 1984)

$$\epsilon_{ij}(l \neq 0) = \frac{(-1)^l}{l!}\frac{1}{T_\perp^{l+1}}\frac{\partial^l}{\partial (T_\perp^{-1})^l}[T_\perp \epsilon_{ij}(l=0)] \tag{20b}$$

where $\epsilon_{ij}(l \neq 0)$ denotes the dielectric tensor for the distribution (20a) with loss-cone index l, while $\epsilon_{ij}(l=0)$ is the dielectric tensor for the bi-Maxwellian distribution (6). Note that it is the derivative with respect to the perpendicular temperature $T_\perp$ that occurs in (20b); this is connected with the fact that the loss-cone in the distribution affects the perpendicular component of particle motion.

3. Relevant limiting cases

Let us now consider a few relevant limiting cases of the dielectric tensor (16).

3.1. *The isotropic limit* $T_\perp = T_\parallel (= T)$

In this case the dielectric tensor (16a–c) reduces to

$$\epsilon_{ij} = \delta_{ij} - \left(\frac{\omega_p}{\omega}\right)^2 \frac{mc^2}{T} \sum_n Q_{ij}^{(n)}(z_n(T), a_\parallel, \lambda_\perp), \tag{21}$$

with

$$Q_{ij}^{(n)} \equiv \begin{pmatrix} \frac{n^2}{\lambda_\perp} R_{\frac{3}{2}}^{(n)} & -in R_{\frac{5}{2}}^{\prime(n)} & -n\left(\frac{mc^2}{T}\frac{N_\parallel^2}{\lambda_\perp}\right)^{\frac{1}{2}} \frac{\partial}{\partial z_n} R_{\frac{3}{2}}^{(n)} \\ in R_{\frac{5}{2}}^{\prime(n)} & \frac{n^2}{\lambda_\perp} R_{\frac{3}{2}}^{(n)} - 2\lambda_\perp R_{\frac{5}{2}}^{\prime(n)} & -i\left(\frac{mc^2}{T} N_\parallel^2 \lambda_\perp\right)^{\frac{1}{2}} \frac{\partial}{\partial z_n} R_{\frac{5}{2}}^{\prime(n)} \\ -n\left(\frac{mc^2}{T}\frac{N_\parallel^2}{\lambda_\perp}\right)^{\frac{1}{2}} \frac{\partial}{\partial z_n} R_{\frac{3}{2}}^{(n)} & i\left(\frac{mc^2}{T} N_\parallel^2 \lambda_\perp\right)^{\frac{1}{2}} \frac{\partial}{\partial z_n} R_{\frac{5}{2}}^{\prime(n)} & \frac{\partial}{\partial N_\parallel}\left(N_\parallel R_{\frac{3}{2}}^{(n)}\right) \end{pmatrix}, \tag{22}$$

where function $R_p^{(n)}$ is the isotropic limit of the function defined by (7) (Robinson 1987):

$$R_p^{(n)}(z_n, a_\parallel, \lambda_\perp) \equiv -i\int_0^\infty d\tau \frac{\exp[iz_n(T)\tau - a_\parallel \tau^2/(1-i\tau)]}{(1-i\tau)^p} \Gamma_n\left(\frac{\lambda_\perp}{1-i\tau}\right), \tag{23}$$

and $R_p^{\prime(n)}$ is given by (23) with Γ_n replaced by Γ_n'. The result (22) coincides with the result given by equations (2.3.78) and (2.3.53) of Bornatici *et al.* (1983a), where the matrix (2.3.53) should be corrected as follows: (i) replace $2(\lambda\rho)^{\frac{1}{2}}$ by

$(\rho/\lambda)^{\frac{1}{2}}$ in the xz (and zx) element; (ii) replace Γ_n by Γ'_n in the yz (and zy) element; (iii) replace ρ by $\rho/x(\tau)$.

3.2. *The non-relativistic limit*

This corresponds to making the following replacements

$$\begin{Bmatrix} R_{q,l}^{(n)} \\ \dfrac{\partial}{\partial z_\parallel} R_{q,l}^{(n)} \\ \dfrac{\partial}{\partial N_\parallel} R_{q,l}^{(n)} \end{Bmatrix} \to -\frac{1}{2a_\parallel^{\frac{1}{2}}} \begin{Bmatrix} Z\left(\dfrac{z_n(T_\parallel)}{2a_\parallel^{\frac{1}{2}}}\right) \\ \dfrac{1}{2a_\parallel^{\frac{1}{2}}} Z'\left(\dfrac{z_n(T_\parallel)}{2a_\parallel^{\frac{1}{2}}}\right) \\ -\dfrac{1}{N_\parallel}\left[Z\left(\dfrac{z_n(T_\parallel)}{2a_\parallel^{\frac{1}{2}}}\right) + \dfrac{z_n(T_\parallel)}{2a_\parallel^{\frac{1}{2}}} Z'\left(\dfrac{z_n(T_\parallel)}{2a_\parallel^{\frac{1}{2}}}\right)\right] \end{Bmatrix} \Gamma_n(\lambda_\perp), \quad (24)$$

where $Z'(x) \equiv dZ/dx$, the same relations also applying for $R_{q,l}^{\prime(n)}$ with $\Gamma_n(\lambda_\perp) \to \Gamma'_n(\lambda_\perp)$. With (24), the dielectric tensor (16a–c) reduces to the well-known non-relativistic expression for a bi-Maxwellian distribution (Melrose 1980).

3.3. *To the lowest significant order in the FLR effects*

Keeping only the contribution from $k = 0$ to the nth harmonic, as well as the $k = 0$ and $k = 1$ contributions to the $n = 0$ terms of ϵ_{zz} and ϵ_{yz} respectively, the dielectric tensor (18) reduces to

$$\begin{Bmatrix} \epsilon_{xx} - 1 = \epsilon_{yy} - 1 \\ \epsilon_{xy} = -\epsilon_{yx} \end{Bmatrix} = \begin{Bmatrix} -1 \\ i \end{Bmatrix} \left(\frac{\omega_p}{\omega}\right)^2 \frac{mc^2}{T_\parallel} \sum_{n=1}^{\infty} \frac{n^2 \lambda_\perp^{n-1}}{2^n n!} \left[W_{\frac{1}{2}, n+1}^{(\pm)} + N_\parallel^2 \left(1 - \frac{T_\perp}{T_\parallel}\right) \frac{\partial}{\partial z_\parallel} W_{\frac{3}{2}, n+1}^{(\mp)} \right], \quad (25a)$$

$$\begin{Bmatrix} \epsilon_{xz} = \epsilon_{zx} \\ \epsilon_{yz} = -\epsilon_{zy} \end{Bmatrix} = \begin{Bmatrix} 1 \\ i \end{Bmatrix} \left(\frac{\omega_p}{\omega}\right)^2 \frac{\omega}{\omega_c} |N_\parallel| N_\perp \frac{T_\perp}{T_\parallel} \frac{mc^2}{T_\parallel} \left[\begin{Bmatrix} 0 \\ -\left(\dfrac{\partial}{\partial z_\parallel} W_{\frac{3}{2}, 2}\right)_{n=0} \end{Bmatrix} \right.$$

$$\left. + \sum_{n=1}^{\infty} \frac{n\lambda_\perp^{n-1}}{2^n n!} \left[\frac{\partial}{\partial z_\parallel} W_{\frac{3}{2}, n+1}^{(\pm)} - \frac{n\omega_c}{\omega}\left(1 - \frac{T_\parallel}{T_\perp}\right) \frac{\partial}{\partial z_\parallel} W_{\frac{3}{2}, n+1}^{(\mp)} \right] \right], \quad (25b)$$

$$\epsilon_{zz} = 1 - \left(\frac{\omega_p}{\omega}\right)^2 \frac{mc^2}{T_\parallel} \left\{ \left[\frac{\partial}{\partial N_\parallel} (N_\parallel W_{\frac{3}{2}, 1}) \right]_{n=0} + \sum_{n=1}^{\infty} \frac{\lambda_\perp^n}{2^n n!} \right.$$

$$\left. \times \left[\frac{\partial}{\partial N_\parallel} (N_\parallel W_{\frac{3}{2}, n+1}^{(+)}) - \frac{n\omega_c}{\omega}\left(1 - \frac{T_\parallel}{T_\perp}\right) \frac{\partial}{\partial N_\parallel} (N_\parallel W_{\frac{3}{2}, n+1}^{(-)}) \right] \right\}. \quad (25c)$$

The results (25a–c) reduce to those obtained by Bornatici *et al.* (1983a) in the isotropic limit, except the $n = 0$ contribution to ϵ_{yz}.

3.4. *Parallel propagation* ($N_\perp = 0$)

For propagation parallel to the magnetic field, $N_\perp = 0$ and thus $\lambda_\perp = 0$, so that all FLR effects vanish. As a consequence, from (25a–c) one obtains

$$\begin{Bmatrix} \epsilon_{xx} - 1 = \epsilon_{yy} - 1 \\ \epsilon_{xy} = -\epsilon_{yx} \end{Bmatrix} = \begin{Bmatrix} -1 \\ i \end{Bmatrix} \left(\frac{\omega_p}{\omega}\right)^2 \frac{mc^2}{T_\parallel} \frac{1}{2} \left[W_{\frac{1}{2}, 2}^{(\pm)} + N_\parallel^2 \left(1 - \frac{T_\perp}{T_\parallel}\right) \frac{\partial}{\partial z_\parallel} W_{\frac{3}{2}, 2}^{(\mp)} \right]_{n=1}, \quad (26a)$$

$$\epsilon_{xz} = \epsilon_{yz} = 0, \quad (26b)$$

$$\epsilon_{zz} = 1 - \left(\frac{\omega_p}{\omega}\right)^2 \frac{mc^2}{T_\parallel} \left[\frac{\partial}{\partial N} (N_\parallel W_{\frac{3}{2}, 1}) \right]_{n=0}. \quad (26c)$$

3.5. *Perpendicular propagation* ($N_\parallel = 0$)

In this case the effect of the temperature anisotropy combined with $N_\parallel^2$ vanishes, and from (18) one gets

$$\begin{Bmatrix} \epsilon_{xx}-1 \\ \epsilon_{xy} = -\epsilon_{yx} \end{Bmatrix} = -\left(\frac{\omega_p}{\omega}\right)^2 \frac{mc^2}{T_\parallel} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \begin{Bmatrix} n^2 \\ -in(k+n) \end{Bmatrix} a_{k,n} \lambda_\perp^{k+n-1} F^{(\pm)}_{\frac{1}{2},k+n+1}, \quad (27a)$$

$$\epsilon_{yy} = 1-\left(\frac{\omega_p}{\omega}\right)^2 \frac{mc^2}{T_\parallel}\left\{\sum_{k=2}^{\infty} \frac{2(k-1)k^2}{2k-1} a_{k,0}\lambda_\perp^{k-1}(F_{\frac{1}{2},k+1})_{n=0}\right.$$

$$\left. + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\left[(k+n)^2 - \frac{k(k+2n)}{2k+2n-1}\right] a_{k,n}\lambda_\perp^{k+n-1} F^{(+)}_{\frac{1}{2},k+n+1}\right\}, \quad (27b)$$

$$\epsilon_{xz} = \epsilon_{yz} = 0, \quad (27c)$$

$$\epsilon_{zz} = 1-\left(\frac{\omega_p}{\omega}\right)^2\left(1+\frac{mc^2}{T_\perp}\sum_{n=1}^{\infty}\frac{n\omega_c}{\omega}\sum_{k=0}^{\infty} a_{k,n}\lambda_\perp^{k+n} F^{(-)}_{\frac{3}{2},k+n+1}\right), \quad (27d)$$

where $F^{(\pm)}_{q,p}$ is defined as in (19), with $W_{q,p}$ replaced by the dispersion function

$$F_{q,p}\left(z_n(T_\parallel), \frac{T_\perp}{T_\parallel}\right) \equiv -i\int_0^\infty d\tau \frac{\exp[{}^{iz_n(T_\parallel)\tau}]}{(1-i\tau)^q(1-i\tau T_\perp/T_\parallel)^p}, \quad (28)$$

which is a generalization to the case of temperature anisotropy of the Dnevstrovskij dispersion function (Dnevstrovskij, Kostomarov & Shrydlov 1964) and contains the explicit effect of the temperature anisotropy. In particular, it should be noted that the imaginary part of the function (28), which exists for $z_n < 0$, i.e. for $\omega \leqslant n\omega_c$, can be expressed in terms of the confluent hypergeometric function (Bornatici *et al.* 1983*b*). For the form of ϵ_{zz} given in (27*a*–*d*), the limit $N_\parallel = 0$ is more easily obtained starting from (1) rather than from (18*a*–*d*).

4. Limiting cases of strong temperature anisotropy

It is of interest to examine two limiting cases of strong temperature anisotropy, namely the case for which $T_\perp \to 0$ and $T_\parallel$ is finite, and vice versa, i.e. $T_\parallel \to 0$ and $T_\perp$ is finite.

With $T_\perp \to 0$, FLR effects vanish and the corresponding expressions for the dielectric tensor are obtained from (25*a*–*c*) by keeping only the $n = 0$ and $n = 1$ terms, i.e.

$$\begin{Bmatrix} \epsilon_{xx}-1 = \epsilon_{yy}-1 \\ \epsilon_{xy} = -\epsilon_{yx} \end{Bmatrix} = \begin{Bmatrix} -1 \\ i \end{Bmatrix}\left(\frac{\omega_p}{\omega}\right)^2 \frac{mc^2}{T_\parallel}\frac{1}{2}\left(W^{(\pm)}_{\frac{1}{2}} + N_\parallel^2 \frac{\partial}{\partial z_\parallel} W^{(\mp)}_{\frac{1}{2}}\right)_{n=1}, \quad (29a)$$

$$\begin{Bmatrix} \epsilon_{xz} = \epsilon_{zx} \\ \epsilon_{yz} = -\epsilon_{zy} \end{Bmatrix} = \begin{Bmatrix} 1 \\ i \end{Bmatrix}\left(\frac{\omega_p}{\omega}\right)^2 \tfrac{1}{2}|N_\parallel| N_\perp \frac{mc^2}{T_\parallel}\left(\frac{\partial}{\partial z_\parallel} W^{(\mp)}_{\frac{3}{2}}\right)_{n=1}, \quad (29b)$$

$$\epsilon_{zz} = 1-\left(\frac{\omega_p}{\omega}\right)^2\left\{\frac{mc^2}{T_\parallel}\left[\frac{\partial}{\partial N_\parallel}(N_\parallel W_{\frac{3}{2}})\right]_{n=0} + \frac{1}{2}\frac{\omega}{\omega_c}N_\perp^2\left[\frac{\partial}{\partial N_\parallel}(N_\parallel W^{(-)}_{\frac{3}{2}})\right]_{n=1}\right\}, \quad (29c)$$

the functions W_q being defined by (10*b*).

To consider the limit $T_\| \to 0$, it is convenient to make the change of variable $t = \tau T_\perp / T_\|$ in the dispersion function (10*a*), so that, as $T_\| \to 0$,

$$W_{q,p} \to \frac{T_\|}{T_\perp} \tilde{F}_p(z_n(T_\perp)), \tag{30}$$

where
$$\tilde{F}_p(z_n(T_\perp)) \equiv -i \int_0^\infty dt \frac{\exp[iz_n(T_\perp)t]}{(1-it)^p}, \tag{31}$$

with p a positive integer and $z_n(T_\perp) = \mu_\perp(1 - n\omega_c/\omega)$. The function $\tilde{F}_p$, which is independent of $N_\|$, is the extension of the Dnevstrovskij function to integer order (Lazzaro & Orefice 1980). With (30), one obtains from (19)

$$\left\{ \begin{matrix} \epsilon_{xx} - 1 \\ \epsilon_{xy} = -\epsilon_{yx} \end{matrix} \right\} = -\left(\frac{\omega_p}{\omega}\right)^2 \frac{mc^2}{T_\perp} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \left\{ \begin{matrix} n^2 \\ -in(k+n) \end{matrix} \right\} \times a_{k,n} \lambda_\perp^{k+n-1} \left(\tilde{F}^{(\pm)}_{k+n+1} - N_\|^2 \frac{\partial}{\partial z_\perp} \tilde{F}^{(\mp)}_{k+n+1} \right), \tag{32a}$$

$$\epsilon_{yy} = 1 - \left(\frac{\omega_p}{\omega}\right)^2 \frac{mc^2}{T_\perp} \left\{ \sum_{k=2}^{\infty} \frac{2(k-1)k^2}{2k-1} a_{k,0} \lambda_\perp^{k-1} \left[\left(1 - N_\|^2 \frac{\partial}{\partial z_\perp}\right) \tilde{F}_{k+1} \right]_{n=0} + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \right. \times \left[(k+n)^2 - \frac{k(k+2n)}{2k+2n-1} \right] a_{k,n} \lambda_\perp^{k+n-1} \left[\tilde{F}^{(+)}_{k+n+1} - N_\|^2 \frac{\partial}{\partial z_\perp} \tilde{F}^{(-)}_{k+n+1} \right], \tag{32b}$$

$$\left\{ \begin{matrix} \epsilon_{xz} = \epsilon_{zx} \\ \epsilon_{yz} = -\epsilon_{zy} \end{matrix} \right\} = \left(\frac{\omega_p}{\omega}\right)^2 \frac{\omega}{\omega_c} |N_\|| N_\perp \frac{mc^2}{T_\perp} \left[\left\{ \begin{matrix} 0 \\ i \sum_{k=1}^{\infty} k a_{k,0} \lambda_\perp^{k-1} \left(\frac{\partial}{\partial z_\perp} \tilde{F}_{k+1} \right)_{n=0} \end{matrix} \right\} + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \left\{ \begin{matrix} n \\ i(k+n) \end{matrix} \right\} a_{k,n} \lambda_\perp^{k+n-1} \left(\frac{\partial}{\partial z_\perp} \tilde{F}^{(\pm)}_{k+n+1} - \frac{n\omega_c}{\omega} \frac{\partial}{\partial z_\perp} \tilde{F}^{(\mp)}_{k+n+1} \right) \right], \tag{32c}$$

$$\epsilon_{zz} = 1 - \left(\frac{\omega_p}{\omega}\right)^2. \tag{32d}$$

The salient characteristic of the dielectric tensor (32*a*–*d*) is the fact that ϵ_{zz} coincides with the cold expression; for the fully relativistic case one would have $\epsilon_{zz} = 1 - (\mu_\perp/(1+\mu_\perp))(\omega_p/\omega)^2$ (Bornaticiti *et al.* 1983*a*).

For the case of strong temperature anisotropy it is possible to obtain the anti-Hermitian part of the dielectric tensor in the fully relativistic form. We consider two cases of strongly anisotropic distribution functions, the first of which is the two-dimensional relativistic Maxwellian (Dawson 1981; Bornatici *et al.* 1983*a*)

$$f_0 = C_2 \exp\left\{ -\mu_\perp \left[1 + \left(\frac{p_\perp}{mc}\right)^2 \right]^{\frac{1}{2}} \right\} \delta(p_\|), \tag{33}$$

where $C_2 \equiv \mu_\perp^2 e^{\mu_\perp} [2\pi(mc)^2(1+\mu_\perp)]^{-1}$. The particle energy connected with (33) is totally in the direction perpendicular to the magnetic field, the corresponding average kinetic energy

$$\langle W \rangle = \left(\int d^3p \gamma f_0 - 1 \right) mc^2$$

being
$$\langle W\rangle = \frac{\mu_\perp+2}{\mu_\perp(\mu_\perp+1)}mc^2, \tag{34}$$
with $\langle W\rangle \approx T_\perp$ to lowest order in $\mu_\perp$ ($= mc^2/T_\perp \gg 1$). With (33), in particular, it is possible to evaluate the anti-Hermitian part of the dielectric tensor (5) exactly, the relevant two-dimensional integration occurring in (5) being carried out by means of the two δ functions occurring. Note that the relevant resonance condition is $[1+(p_\perp/mc)^2]^{\frac{1}{2}} - n\omega_c/\omega = 0$, $n \geqslant 1$, which is meaningful only if relativistic effects are taken into account. Explicitly, using (B 12)–(B 14) from Appendix B, one finds

$$\begin{Bmatrix}\epsilon_{a,xx}\\ \epsilon_{a,xy}=-\epsilon_{a,yx}\\ \epsilon_{a,yy}\end{Bmatrix} = \pi\left(\frac{\omega_p}{\omega}\right)^2\frac{\mu_\perp^2}{\mu_\perp+1}\frac{1}{\lambda_\perp}\sum_{n=1}^{\infty}\left[(1-N_\parallel^2)\begin{Bmatrix}n^2J_n^2\\ -inb_nJ_nJ'_n\\ b_n^2J'^2_n\end{Bmatrix}\right.$$
$$\left.+N_\parallel^2\frac{n\omega_c}{\omega}\lambda_\perp\begin{Bmatrix}2\dfrac{n^2J_nJ'_n}{b_n}\\ -in\left[\left(\dfrac{n^2}{b_n^2}-1\right)J_n^2+J'^2_n\right]\\ 2b_n\left(\dfrac{n^2}{b_n^2}-1\right)J_nJ'_n\end{Bmatrix}\right]\exp\left[-\mu_\perp\left(\frac{n\omega_c}{\omega}-1\right)\right], \tag{35a}$$

$$\begin{Bmatrix}\epsilon_{a,xz}=\epsilon_{a,zx}\\ \epsilon_{a,yz}=-\epsilon_{a,zy}\end{Bmatrix} = \pi\left(\frac{\omega_p}{\omega}\right)^2\frac{\omega_c}{\omega}\frac{\mu_\perp^2}{\mu_\perp+1}\frac{N_\parallel}{N_\perp}\sum_{n=1}^{\infty}\begin{Bmatrix}nJ_n^2\\ ib_nJ_nJ'_n\end{Bmatrix}\exp\left[-\mu_\perp\left(\frac{n\omega_c}{\omega}-1\right)\right], \tag{35b}$$

$$\epsilon_{a,zz} = 0, \tag{35c}$$
where $J_n \equiv J_n(b_n)$, $J'_n \equiv dJ_n/db_n$ and $b_n \equiv N_\perp[n^2-(\omega/\omega_c)^2]^{\frac{1}{2}}$, $\omega \leqslant n\omega_c$.

Note that the salient characteristic $\epsilon_{a,zz} = 0$ is a consequence of the absence of parallel motion of the two-dimensional distribution (33); also, $\epsilon_{a,ij} = 0$ for $\omega = n\omega_c$ and $N_\parallel = 0$.

(*a*) In the limit of propagation *perpendicular* ($N_\parallel = 0$) to the magnetic field one has
$$\left.\begin{aligned}&\epsilon_{a,ij}(N_\parallel=0) = \sum_{n=1}^{\infty}\epsilon_{a,ij}^{(n)},\\ &\text{with}\quad \epsilon_{a,xy}^{(n)} = -i\frac{b_nJ'_n}{nJ_n}\epsilon_{a,xx}^{(n)},\quad \epsilon_{a,yy}^{(n)} = \left(\frac{b_nJ'_n}{nJ_n}\right)^2\epsilon_{a,xx}^{(n)},\\ &\epsilon_{a,xx}^{(n)} = \pi\left(\frac{\omega_p}{\omega}\right)^2\frac{\mu_\perp^2}{\mu_\perp+1}\frac{n^2J_n^2}{\lambda_\perp}\exp\left[-\mu_\perp\left(\frac{n\omega_c}{\omega}-1\right)\right],\end{aligned}\right\} \tag{36}$$
$$\epsilon_{a,iz} = 0 \quad \text{for } i = x, y, z.$$

(*b*) In the limit of propagation *parallel* ($N_\perp = 0$) to the magnetic field. FLR effects have to be kept to lowest order, and one obtains
$$\left.\begin{aligned}&\epsilon_{a,xy} = -i\epsilon_{a,xx},\quad \epsilon_{a,yy} = \epsilon_{a,xx},\\ &\epsilon_{a,xx} = \frac{\pi}{2}\left(\frac{\omega_p}{\omega}\right)^2\frac{\mu_\perp^2}{\mu_\perp+1}\left\{\tfrac{1}{2}(1-N_\parallel^2)\mu_\perp\left[\left(\frac{\omega_c}{\omega}\right)^2-1\right]+\frac{N_\parallel^2\omega_c}{\omega}\right\}\exp\left[-\mu_\perp\left(\frac{\omega_c}{\omega}-1\right)\right],\end{aligned}\right\} \tag{37}$$

for which only the first harmonic $n = 1$ contributes. Furthermore, $\epsilon_{a,iz} = 0$ for $i = x, y, z$.

Another example of distribution with strong temperature anisotropy is the one-dimensional relativistic Maxwellian

$$f_0 = C_1 \exp\left\{-\mu_\parallel\left[1+\left(\frac{p_\parallel}{mc}\right)^2\right]^{\frac{1}{2}}\right\}\frac{\delta(p_\perp)}{2\pi p_\perp}, \tag{38}$$

where $C_1 \equiv [2mcK_1(\mu_\parallel)]^{-1}$, K_n denoting the modified Bessel function of the second kind of order n. In this case the particle motion is along the magnetic field; thus, in particular, no FLR effects are present, and the average (parallel) kinetic energy is

$$\langle W\rangle = \left[\frac{K_0(\mu_\parallel)}{K_1(\mu_\parallel)} - 1 + \frac{1}{\mu_\parallel}\right]mc^2, \tag{39}$$

so that $\langle W\rangle \approx \frac{1}{2}T_\parallel$, to lowest order in $\mu_\parallel\ (= mc^2/T_\parallel \gg 1)$. With (38), using (B 20) and (B 23) from Appendix B, one obtains for the anti-Hermitian part of the dielectric tensor (5)

$$\epsilon_{a,xx}\ (= \epsilon_{a,yy} = i\epsilon_{a,xy}) = \tfrac{1}{2}\pi a(\mu_\parallel)\left(\frac{\omega_p}{\omega}\right)^2 \frac{\omega_c}{\omega}\mu_\parallel^{\frac{3}{2}}\frac{|N_\parallel|}{|1-N_\parallel^2|}\frac{I_{-\frac{1}{2}}(x)}{x^{\frac{1}{2}}}$$
$$\times \exp\left[-\mu_\parallel\left(\frac{\omega_c/\omega}{1-N_\parallel^2}-1\right)\right], \tag{40a}$$

$$\epsilon_{a,xz}\ (= -i\epsilon_{a,yz}) = \tfrac{1}{2}\pi a(\mu_\parallel)\left(\frac{\omega_p}{\omega}\right)^2 \mu_\parallel^{\frac{1}{2}}\frac{N_\perp}{|1-N_\parallel^2|}\left[\frac{\mu_\parallel N_\parallel^2}{1-N_\parallel^2}\frac{\omega_c}{\omega}\frac{I_{-\frac{1}{2}}(x)}{x^{\frac{1}{2}}} - x^{\frac{1}{2}}I_{\frac{1}{2}}(x)\right]$$
$$\times \exp\left[-\mu_\parallel\left(\frac{\omega_c/\omega}{1-N_\parallel^2}-1\right)\right], \tag{40b}$$

$$\epsilon_{a,zz} = (\tfrac{1}{2}\pi)^{\frac{1}{2}}a(\mu_\parallel)\left(\frac{\omega_p}{\omega}\right)^2\left(\frac{\mu_\parallel}{N_\parallel^2-1}\right)^{\frac{3}{2}}\exp\left\{-\mu_\parallel\left[\frac{|N_\parallel|}{(N_\parallel^2-1)^{\frac{1}{2}}}-1\right]\right\}$$
$$+\tfrac{1}{2}\pi a(\mu_\parallel)\left(\frac{\omega_p}{\omega}\right)^2\mu_\parallel^{\frac{1}{2}}\frac{N_\perp^2|N_\parallel|}{(1-N_\parallel^2)^2}\left[\left(\frac{\mu_\parallel N_\parallel^2}{1-N_\parallel^2}\frac{\omega_c}{\omega}+\frac{|1-N_\parallel^2|}{\mu_\parallel N_\parallel^2}\frac{\omega}{\omega_c}x^2\right)\frac{I_{-\frac{1}{2}}(x)}{x^{\frac{1}{2}}} - 2x^{\frac{1}{2}}I_{\frac{1}{2}}(x)\right]$$
$$\times \exp\left[-\mu_\parallel\left(\frac{\omega_c/\omega}{1-N_\parallel^2}-1\right)\right], \tag{40c}$$

where $a(\mu_\parallel) \equiv (\pi/2\mu_\parallel)^{\frac{1}{2}}e^{-\mu_\parallel}/K_1(\mu_\parallel)$ ($=1$ to lowest order in $\mu_\parallel^{-1}$),

$$x \equiv \frac{\mu_\parallel|N_\parallel|}{|1-N_\parallel^2|}\left[N_\parallel^2+\left(\frac{\omega_c}{\omega}\right)^2-1\right]^{\frac{1}{2}}, \quad N_\parallel^2+\left(\frac{\omega_c}{\omega}\right)^2-1>0. \tag{41}$$

It should be noted that (i) $\epsilon_{a,ij} = 0$ when the inequality in (41) is reversed; (ii) only the contribution from the harmonic $n = 1$ is considered explicitly, the corresponding contribution from $n = -1$ being obtained with $\omega_c \rightarrow -\omega_c$ appropriately; (iii) the first term on the right-hand side of $\epsilon_{a,zz}$ is connected with the Čerenkov resonance ($n = 0$), which only exists for $N_\parallel^2 > 1$; and (iv) $\epsilon_{a,xx}$ is independent of $N_\perp$, whereas $\epsilon_{a,xz}$ and the contribution to $\epsilon_{a,zz}$ due to cyclotron resonance are proportional to $N_\perp$ and $N_\perp^2$ respectively.

In the limit of *perpendicular propagation* ($N_\parallel = 0$), by using the series expansion of the Bessel functions $I_{\pm\frac{1}{2}}$, one obtains

$$\left.\begin{aligned}
\epsilon_{a,xx}(N_\parallel = 0) &= (\tfrac{1}{2}\pi)^{\frac{1}{2}} a(\mu_\parallel)\left(\frac{\omega_p}{\omega}\right)^2 \frac{\omega_c}{\omega}\mu_\parallel^{\frac{1}{2}}\left[\left(\frac{\omega_c}{\omega}\right)^2 - 1\right]^{-\frac{1}{2}} \exp\left[-\mu_\parallel\left(\frac{\omega_c}{\omega} - 1\right)\right],\\
\epsilon_{a,xz}(N_\parallel = 0) &= 0,\\
\epsilon_{a,zz}(N_\parallel = 0) &= N_\perp^2\left[1 - \left(\frac{\omega}{\omega_c}\right)^2\right]\epsilon_{a,xx}(N_\parallel = 0)
\end{aligned}\right\} \quad (42)$$

with $\omega < \omega_c$.

5. Conclusions

The dielectric tensor relevant to the electron-cyclotron interaction in an anisotropic plasma has been evaluated for a bi-Maxwellian distribution function, in the weakly relativistic approximation and to any order in finite-Larmor-radius effects. A number of relevant limiting cases have been considered, including two cases of strong temperature anisotropy.

In particular, the anti-Hermitian part of the (fully relativistic) dielectric tensor has been obtained for a two-dimensional as well as one-dimensional (relativistic) Maxwellian distribution.

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Appendix A. Integrals relevant to the evaluation of the dielectric tensor (1)

Within the weakly relativistic approximation, i.e. $\gamma \approx 1 + (p_\perp^2 + p_\parallel^2)/2m^2c^2$, the integrations with respect to the perpendicular, $p_\perp$, and parallel, $p_\parallel$, momenta that occur in the dielectric tensor (1) are independent of each other provided that the equilibrium distribution function is of the form $f_0(p_\perp, p_\parallel) = g(p_\perp)\,h(p_\parallel)$. For the specific case of the bi-Maxwellian distribution (6), the analytical evaluation of the $p_\perp$ and $p_\parallel$ integrals proceeds along the same lines as for the non-relativistic approximation (Melrose 1980). More specifically, the relevant integrals with respect to $p_\parallel\,(= (2mT_\parallel)^{\frac{1}{2}}x)$ are

$$\int_{-\infty}^{\infty} dx \exp\left[-(1-i\tau)\,x^2 - 2ia_\parallel^{\frac{1}{2}}\tau x\right]\begin{Bmatrix}1\\ x\\ x^2\end{Bmatrix} = \frac{\pi^{\frac{1}{2}}}{(1-i\tau)^{\frac{1}{2}}}$$

$$\times\begin{Bmatrix}1\\ -\dfrac{ia_\parallel^{\frac{1}{2}}\tau}{1-i\tau}\\ \dfrac{1}{2(1-i\tau)}\left(1 + N_\parallel^2\dfrac{mc^2}{T_\parallel}\dfrac{\partial}{\partial a_\parallel}\right)\end{Bmatrix}\exp\left(-\frac{a_\parallel\tau^2}{1-i\tau}\right), \quad \text{(A 1)}$$

with $a_\parallel \equiv \frac{1}{2}N_\parallel^2 mc^2/T_\parallel$ and $\tau \equiv (T_\parallel/mc^2)\,t$, t being the same integration variable that occurs in (1). As $1 - i\tau \to 1$, one recovers the non-relativistic limit.

For the integration over $p_\perp$ ($= (2mT_\perp)^{\frac{1}{2}}(1-i\tau T_\perp/T_\parallel)^{-\frac{1}{2}}y$), one has the following integrals:

$$\int_0^\infty dy\, y\, e^{-y^2}\begin{Bmatrix} J_n^2(\tilde{b}y) \\ yJ_n(\tilde{b}y)\,J_n'(\tilde{b}y) \\ y^2 J_n'^2(\tilde{b}y)\end{Bmatrix} = \frac{1}{2}\begin{Bmatrix} \Gamma_n(\Lambda_\perp) \\ (\tfrac{1}{2}\Lambda_\perp)^{\frac{1}{2}}\Gamma_n'(\Lambda_\perp) \\ \dfrac{1}{2\Lambda_\perp}[n^2\Gamma_n(\Lambda_\perp) - 2\Lambda_\perp^2\Gamma_n'(\Lambda_\perp)]\end{Bmatrix}, \quad \begin{matrix}(\mathrm{A}\ 2)\\ (\mathrm{A}\ 3)\\ (\mathrm{A}\ 4)\end{matrix}$$

where $\tilde{b} \equiv (2\Lambda_\perp)^{\frac{1}{2}}$, $\Gamma_n(x) \equiv e^{-x}I_n(x)$, $\Lambda_\perp = \lambda_\perp/(1-i\tau T_\perp/T_\parallel)$ and $\lambda_\perp = (N_\perp\,\omega/\omega_c)^2\,T_\perp/mc^2$. As $\Lambda_\perp \to \lambda_\perp$, (A 2)–(A 4) reduce to the non-relativistic results (Melrose 1980). Using (A 1)–(A 4) in (1) yields the forms (16*a–c*) for the dielectric tensor for the bi-Maxwellian distribution.

For practical purposes it is useful to express the results (A 2)–(A 4) in terms of a series representation with respect to the Larmor parameter $\Lambda_\perp$. This can be accomplished by replacing J_n^2, $J_n J_n'$ and $J_n'^2$ in (A 2)–(A 4) by their series representations and carrying out the corresponding integrations term by term. More specifically, noting that (Gradshteyn & Ryzhik 1980)

$$J_n^2(z) = \sum_{k=0}^{\infty}\frac{(-1)^k[2(|n|+k)]!}{[(|n|+k)!]^2(2|n|+k)!\,k!}(\tfrac{1}{2}z)^{2(|n|+k)}, \tag{A 5}$$

from (A 2) one finds

$$\Gamma_n(\Lambda_\perp) = (\tfrac{1}{2}\Lambda_\perp)^{|n|}\sum_{k=0}^{\infty}\frac{(-1)^k[2(|n|+k)]!}{(|n|+k)!\,(2|n|+k)!\,k!}(\tfrac{1}{2}\Lambda_\perp)^k. \tag{A 6}$$

Moreover, from (A 5),

$$J_n(z)\,J_n'(z) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k[2(|n|+k)]!}{(|n|+k-1)!\,(|n|+k)!\,(2|n|+k)!\,k!}(\tfrac{1}{2}z)^{2(|n|+k)-1}$$

and together with (A 3) one has

$$\Gamma_n'(\Lambda_\perp) = \tfrac{1}{2}(\tfrac{1}{2}\Lambda_\perp)^{|n|-1}\sum_{k=0}^{\infty}\frac{(-1)^k[2(|n|+k)]!}{(|n|+k-1)!\,2(|n|+k)!\,k!}(\tfrac{1}{2}\Lambda_\perp)^k. \tag{A 7}$$

The series representation of the result given by (A 4) follows simply from (A 6) and (A 7).

Appendix B. Evaluation of the anti-Hermitian part of the dielectric tensor

The salient characteristic of the expression (5) for the anti-Hermitian part of the dielectric tensor is the δ function expressing the (relativistic) cyclotron resonance. To directly exploit the δ function in the integration over the momentum, it is convenient to use the variables $(\gamma, \bar{p}_\parallel)$ instead of $(\bar{p}_\perp, \bar{p}_\parallel)$, where $\bar{p}_{\perp,\parallel} \equiv p_{\perp,\parallel}/mc$. By making the change of variable $\bar{p}_\perp^2 = \gamma^2 - (1+\bar{p}_\parallel^2)$, where $\gamma \geqslant (1+\bar{p}_\parallel^2)^{\frac{1}{2}}$, and carrying out the corresponding γ integration by means of the δ function, one finds from (5)

$$\epsilon_{a,ij} = -2\pi^2(mc)^3\left(\frac{\omega_p}{\omega}\right)^2\sum_{n=-\infty}^{\infty}\int_{\bar{p}_{\parallel,\min}}^{\bar{p}_{\parallel,\max}} d\bar{p}_\parallel[\bar{p}_\perp^2\,V_i^{(n)}(V_j^{(n)})^*]_{\bar{p}_\perp=\bar{p}_\perp^{(\mathrm{res})}}$$
$$\times\left[\left(\frac{\partial}{\partial\gamma}+N_\parallel\frac{\partial}{\partial\bar{p}_\parallel}\right)f_0(\gamma,\bar{p}_\parallel)\right]_{\gamma=N_\parallel\bar{p}_\parallel+n\omega_c/\omega}, \tag{B 1}$$

where

$$\bar{p}_\perp^{(\mathrm{res})} \equiv \left[\left(N_\parallel \bar{p}_\parallel + \frac{n\omega_c}{\omega}\right)^2 - (1+\bar{p}_\parallel^2)\right]^{\frac{1}{2}} = [(N_\parallel^2-1)(\bar{p}_\parallel - \bar{p}_\parallel^{(+)})(\bar{p}_\parallel - \bar{p}_\parallel^{(-)})]^{\frac{1}{2}}, \quad \text{(B 2)}$$

$$\bar{p}_\parallel^{(\pm)} \equiv \frac{N_\parallel}{1-N_\parallel^2}\frac{n\omega_c}{\omega} \pm \frac{1}{|1-N_\parallel^2|}\left[N_\parallel^2 + \left(\frac{n\omega_c}{\omega}\right)^2 - 1\right]^{\frac{1}{2}}. \quad \text{(B 3)}$$

For the limits of the $p_\parallel$ integration in (B 1), for $N_\parallel^2 < 1$ it follows from (B 2) that $\bar{p}_{\parallel,\max} = \bar{p}_\parallel^{(+)}$ and $\bar{p}_{\parallel,\min} = \bar{p}_\parallel^{(-)}$, for which $N_\parallel^2 + (n\omega_c/\omega)^2 - 1 > 0$; for $N_\parallel^2 > 1$, the $p_\parallel$ integration extends over the two intervals $(-\infty, \bar{p}_\parallel^{(-)})$ and $(\bar{p}_\parallel^{(+)}, \infty)$. In (B 1) we have also expressed the quantity $\hat{U}f_0$, given by (3), in terms of the variables γ and $\bar{p}_\parallel$, i.e.

$$\hat{U}f_0 = \frac{1}{mc}\frac{\bar{p}_\perp}{\gamma}\left(\frac{\partial}{\partial\gamma} + N_\parallel \frac{\partial}{\partial \bar{p}_\parallel}\right) f_0(\gamma, \bar{p}_\parallel). \quad \text{(B 4)}$$

The following points are worth noting.

(i) For

$$N_\parallel^2 \gg \left|1 - \left(\frac{n\omega_c}{\omega}\right)^2\right| \quad \text{(B 5)}$$

(B 3) yields the non-relativistic resonant value

$$\bar{p}_\parallel = \frac{1}{N_\parallel}\left(1 - \frac{n\omega_c}{\omega}\right), \quad \text{(B 6)}$$

along with $\bar{p}_\parallel = [N_\parallel/(1-N_\parallel^2)](1+n\omega_c/\omega)$, the latter value being larger than (B 6) by a factor $2N_\parallel^2|1-n\omega_c/\omega|^{-1}$, typically; both values are such that $\bar{p}_\parallel^2 \ll 1$.

(ii) For $N_\parallel \approx 1$, one finds from (B 3) the two resonant values for $\bar{p}_\parallel$:

$$\bar{p}_\parallel = \frac{1}{2}\frac{\omega}{n\omega_c}\left[1 - \left(\frac{n\omega_c}{\omega}\right)^2\right], \quad \frac{1}{1-N_\parallel}\frac{n\omega_c}{\omega}, \quad \text{(B 7)}$$

the first reducing to (B 6) for $N_\parallel = 1$ and to lowest order in $(1-n\omega_c/\omega)^2$.

(iii) For perpendicular propagation, $N_\parallel = 0$ and (B 1) yields

$$\epsilon_{a,ij}(N_\parallel = 0) = -2\pi^2 (mc)^3 \left(\frac{\omega_p}{\omega}\right)^2 \sum_{n=1}^{\infty}$$

$$\times \int_{\bar{p}_{\parallel,\min}}^{\bar{p}_{\parallel,\max}} d\bar{p}_\parallel [\bar{p}_\perp^2 V_i^{(n)} (V_j^{(n)})^*]_{\bar{p}_\perp = [(n\omega_c/\omega)^2 - 1 - \bar{p}_\parallel^2]^{\frac{1}{2}}} \left[\frac{\partial}{\partial\gamma} f_0(\gamma, \bar{p}_\parallel)\right]_{\gamma = n\omega_c/\omega}, \quad \text{(B 8)}$$

where $\omega < n\omega_c$, $n \geqslant 1$, $\bar{p}_{\parallel,\min} = -[(n\omega_c/\omega)^2 - 1]^{\frac{1}{2}}$ and $\bar{p}_{\parallel,\max} = [(n\omega_c/\omega)^2 - 1]^{\frac{1}{2}}$.

An approximation that is commonly used in the evaluation of the anti-Hermitian part of the dielectric tensor (cf. §2) is the so-called weakly relativistic approximation, which amounts to taking $\gamma \approx 1 + \frac{1}{2}(\bar{p}_\parallel^2 + \bar{p}_\perp^2)$. With such an approximation, (B 1) is still valid with

$$\bar{p}_\perp^{(\mathrm{res})} = [(P^{(+)} - \bar{p}_\parallel)(\bar{p}_\parallel - P^{(-)})]^{\frac{1}{2}} \quad \text{(B 9)}$$

and $\bar{p}_{\parallel,\max} = P^{(+)}$, $\bar{p}_{\parallel,\min} = P^{(-)}$, where $P^{(\pm)}$ denote the two resonant values of $\bar{p}_\parallel$,

$$P^{(\pm)} = N_\parallel \pm \left[N_\parallel^2 + 2\left(\frac{n\omega_c}{\omega} - 1\right)\right]^{\frac{1}{2}}, \quad \text{(B 10)}$$

with $N_\parallel^2+2(n\omega_c/\omega-1)\geqslant 0$. Also, for the quantity within the second set of square brackets in the integrand of (B 1) one should go back to $\hat{U}f_0$, by means of (B 4), and evaluate it at $\bar{p}_\perp=\bar{p}_\perp^{(\text{res})}$ as given by (B 9). As required by the weakly relativistic approximation itself, the solutions (B 10) are subject to the condition $|P^{(\pm)}|^2 \ll 1$; hence, in particular, the solution $P^{(+)}$ is valid for max $\{N_\parallel^2, 2|n\omega_c/\omega-1|\} \ll 1$ and is just the approximation of the (fully relativistic) solution $\bar{p}_\parallel^{(+)}$, (B 3). On the other hand, the solution $P^{(-)}$ is valid for arbitrary $N_\parallel$ and corresponds to either $\bar{p}_\parallel^{(-)}$ or $\bar{p}_\parallel^{(+)}$ of (B 3), depending on whether $N_\parallel^2$ is less than or greater than unity respectively. In other words, to account for the relativistic effects in the resonance condition requires a fully relativistic treatment if $N_\parallel^2 \geqslant 1$, the weakly relativistic approximation being adequate for $N_\parallel^2<1$. The latter is the case that is usually considered (Krivenski & Orefice 1983).

Let us go back to (B 1) and consider a separable distribution function of the form $f_0(\gamma,\bar{p}_\parallel)=g(\gamma)\,h(\bar{p}_\parallel)$. With regard to the second term within square brackets one can perform an integration by parts, with the result

$$\epsilon_{a,ij}=2\pi^2(mc)^3\left(\frac{\omega_p}{\omega}\right)^2\sum_{n=-\infty}^{\infty}\int_{\bar{p}_{\parallel,\min}}^{\bar{p}_{\parallel,\max}} d\bar{p}_\parallel\,h(\bar{p}_\parallel)$$

$$\times\left\{-(1-N_\parallel^2)\,[\bar{p}_\perp^2\,V_i^{(n)}(V_j^{(n)})^*]_{\bar{p}_\perp=\bar{p}_\perp^{(\text{res})}}\left(\frac{dg}{d\gamma}\right)_{\gamma=N_\parallel\bar{p}_\parallel+n\omega_c/\omega}\right.$$

$$\left.+N_\parallel\,g\left(\gamma=N_\parallel\bar{p}_\parallel+\frac{n\omega_c}{\omega}\right)\frac{d}{d\bar{p}_\parallel}[\bar{p}_\perp^2\,V_i^{(n)}(V_j^{(n)})^*]_{\bar{p}_\perp=\bar{p}_\perp^{(\text{res})}}\right\}. \quad \text{(B 11)}$$

For the particular case for which $h(\bar{p}_\parallel)=\delta(\bar{p}_\parallel)$, (B 11) yields

$$\epsilon_{a,ij}=2\pi^2(mc)^3\left(\frac{\omega_p}{\omega}\right)^2\sum_{n=1}^{\infty}\left\{-(1-N_\parallel^2)\,[\bar{p}_\perp^2\,V_i^{(n)}(V_j^{(n)})^*]_{\bar{p}_\perp=[(n\omega_c/\omega)^2-1]^{\frac{1}{2}}}\left(\frac{dg}{d\gamma}\right)_{\gamma=n\omega_c/\omega}\right.$$

$$\left.+N_\parallel\,g\left(\gamma=\frac{n\omega_c}{\omega}\right)\frac{d}{d\bar{p}_\parallel}[\bar{p}_\perp^2\,V_i^{(n)}(V_j^{(n)})^*]_{\bar{p}_\perp=[(n\omega_c/\omega)^2-1]^{\frac{1}{2}}}\right\}_{\bar{p}_\parallel=0}, \quad \text{(B 12)}$$

where, using (2),

$$[\bar{p}_\perp^2\,V_i^{(n)}(V_j^{(n)})^*]_{\substack{\bar{p}_\perp=[(n\omega_c/\omega)^2-1]^{\frac{1}{2}},\\ \bar{p}_\parallel=0}}=\left(\frac{\omega_c}{\omega}\right)^2\frac{1}{N_\parallel^2}\begin{pmatrix} n^2J_n^2 & -inb_n\,J_n\,J_n' & 0\\ inb_n\,J_n\,J_n' & b_n^2(J_n')^2 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad \text{(B 13)}$$

$$\left\{\frac{d}{d\bar{p}_\parallel}[\bar{p}_\perp^2\,V_i^{(n)}(V_j^{(n)})^*]_{\bar{p}_\perp=\bar{p}_\perp^{(\text{res})}}\right\}_{\bar{p}_\parallel=0}=\frac{n\omega_c}{\omega}$$

$$\times\begin{pmatrix} N_\parallel\dfrac{2n^2J_n\,J_n'}{b_n} & -iN_\parallel\,n(J_n\,J_n')' & \dfrac{1}{N_\perp J_n^2}\\ iN_\parallel\,n(J_n\,J_n')' & N_\parallel N_\perp\dfrac{\omega}{\omega_c}2b_n\,J_n'\,J_n'' & \dfrac{i}{N_\perp}\dfrac{b_n}{n}J_n\,J_n'\\ \dfrac{1}{N_\perp}J_n^2 & -\dfrac{i}{N_\perp}\dfrac{b_n}{n}J_n\,J_n' & 0\end{pmatrix}, \quad \text{(B 14)}$$

with $J_n \equiv J_n(b_n)$, $b_n \equiv N_\perp(\omega/\omega_c)[(n\omega_c/\omega)^2-1]^{\frac{1}{2}}$, $\omega \leqslant n\omega_c$. Note that $\epsilon_{a,zz}=0$ as a consequence of the absence of motion along the magnetic field ($f_0 \propto \delta(p_\parallel)$).

Let us now consider the case of a distribution function of the form

$$f_0 = h(\bar{p}_\parallel)\frac{\delta(\bar{p}_\perp)}{2\pi\bar{p}_\perp}.$$

The factor in the integrand of (B 1) containing the derivatives of f_0 can be written as

$$\left(\frac{\partial}{\partial\gamma}+N_\parallel\frac{\partial}{\partial\bar{p}_\parallel}\right)f_0(\gamma,\bar{p}_\parallel)\bigg|_{\gamma=N_\parallel\bar{p}_\parallel+n\omega_c/\omega} = \frac{1}{2\pi}\left[N_\parallel\frac{\delta(\bar{p}_\perp)}{\bar{p}_\perp}\frac{dh}{d\bar{p}_\parallel}+\frac{n\omega_c}{\omega}h(\bar{p}_\parallel)\frac{1}{\bar{p}_\perp}\frac{d}{d\bar{p}_\perp}\frac{\delta(\bar{p}_\perp)}{\bar{p}_\perp}\right]. \quad \text{(B 15)}$$

Furthermore, using (B 2) and (B 3),

$$\frac{\delta(\bar{p}_\perp)}{\bar{p}_\perp}\bigg|_{\bar{p}_\perp=\bar{p}_\perp^{(\text{res})}} = \frac{\delta(\bar{p}_\parallel-\bar{p}_\parallel^{(+)})+\delta(\bar{p}_\parallel-\bar{p}_\parallel^{(-)})}{[N_\parallel^2+(n\omega_c/\omega)^2-1]^{\frac{1}{2}}}, \quad \text{(B 16)}$$

$$\left[\frac{d}{d\bar{p}_\perp}\frac{\delta(\bar{p}_\perp)}{\bar{p}_\perp}\right]_{\bar{p}_\perp=\bar{p}_\perp^{(\text{res})}} = \frac{d\bar{p}_\parallel}{d\bar{p}_\perp^{(\text{res})}}\frac{d}{d\bar{p}_\parallel}\left[\frac{\delta(\bar{p}_\perp)}{\bar{p}_\perp}\right]_{\bar{p}_\perp=\bar{p}_\perp^{(\text{res})}}. \quad \text{(B 17)}$$

Substituting (B 15)–(B 17) into (B 1) and carrying out an integration by parts of the term connected with (B 17) yields

$$\epsilon_{a,ij} = -\pi(mc)^3\left(\frac{\omega_p}{\omega}\right)^2\sum_{n=-\infty}^{\infty}\frac{1}{[N_\parallel^2+(n\omega_c/\omega)^2-1]^{\frac{1}{2}}}\int_{\bar{p}_{\parallel,\min}}^{\bar{p}_{\parallel,\max}}d\bar{p}_\parallel$$
$$\times\left\{N_\parallel\frac{dh}{d\bar{p}_\parallel}[\bar{p}_\perp^2V_i^{(n)}(V_j^{(n)})^*]-\frac{n\omega_c}{\omega}\frac{d}{d\bar{p}_\parallel}\left\{[\bar{p}_\perp^2V_i^{(n)}(V_j^{(n)})^*]_{\bar{p}_\perp=\bar{p}_\perp^{(\text{res})}}h(\bar{p}_\parallel)\frac{1}{\bar{p}_\perp^{(\text{res})}}\frac{d\bar{p}_\parallel}{d\bar{p}_\perp^{(\text{res})}}\right\}\right\}_{\bar{p}_\perp=0}$$
$$\times[\delta(\bar{p}_\parallel-\bar{p}_\parallel^{(+)})+\delta(\bar{p}_\parallel-\bar{p}_\parallel^{(-)})]. \quad \text{(B 18)}$$

With regard to the first term on the right-hand side of (B 18), using (2), one has

$$[\bar{p}_\perp^2V_i^{(n)}(V_j^{(n)})^*]_{\bar{p}_\perp=0} = \bar{p}_\parallel^2\delta_{iz}\delta_{jz} \quad \text{(B 19)}$$

with $n=0$, the corresponding resonant $\bar{p}_\parallel$s being (cf. (B 3)) $\bar{p}_\parallel^{(\pm)}(n=0) = \pm(N_\parallel^2-1)^{-\frac{1}{2}}$ with $N_\parallel^2>1$. For $n=0$ the resonance condition is $\gamma=N_\parallel\bar{p}_\parallel$, so that $\bar{p}_\parallel$ is greater or less than zero, depending on whether $N_\parallel$ is greater or less than zero, the corresponding range of integration over $\bar{p}_\parallel$ being $(\bar{p}_\parallel^{(+)},\infty)$ or $(-\infty,\bar{p}_\parallel^{(-)})$ respectively. Hence the anti-Hermitian part of the dielectric tensor connected with the (relativistic) Čerenkov resonance $n=0$ is

$$\epsilon_{a,ij}(n=0) = -\pi(mc)^3\left(\frac{\omega_p}{\omega}\right)^2\frac{|N_\parallel|}{(N_\parallel^2-1)^{\frac{3}{2}}}\frac{dh}{d\bar{p}_\parallel}\bigg|_{\bar{p}_\parallel=(N_\parallel^2-1)^{-\frac{1}{2}}}\delta_{iz}\delta_{jz}, \quad N_\parallel^2>1. \quad \text{(B 20)}$$

For the second term on the right-hand-side of (B 18) one has

$$\left\{\frac{d}{d\bar{p}_\parallel}\left\{[\bar{p}_\perp^2V_i^{(n)}(V_j^{(n)})^*]_{\bar{p}_\perp=\bar{p}_\perp^{(\text{res})}}h(\bar{p}_\parallel)\frac{1}{\bar{p}_\perp^{(\text{res})}}\frac{d\bar{p}_\parallel}{d\bar{p}_\perp^{(\text{res})}}\right\}\right\}_{\bar{p}_\perp=0}$$
$$= 2[V_i^{(n)}(V_j^{(n)})^*]_{\bar{p}_\perp=0}h(\bar{p}_\parallel), \quad \text{(B 21)}$$

where using (2),

$$[V_i^{(n)}(V_j^{(n)})^*]_{\bar{p}_\perp=0} = \frac{1}{4}\begin{pmatrix} 1 & \mp i & \pm N_\perp \dfrac{\omega}{\omega_c}\bar{p}_\parallel \\ \pm i & 1 & iN_\perp \dfrac{\omega}{\omega_c}\bar{p}_\parallel \\ \pm N_\perp \dfrac{\omega}{\omega_c}\bar{p}_\parallel & -iN_\perp \dfrac{\omega}{\omega_c}\bar{p}_\parallel & \left(N_\perp \dfrac{\omega}{\omega_c}\bar{p}_\parallel\right)^2 \end{pmatrix} \equiv \tfrac{1}{4}H_{ij}(\bar{p}_\parallel, n=\pm 1) \quad \text{(B 22)}$$

for $n = \pm 1$, the upper (lower) signs in (B 22) referring to $n = 1$ $(n = -1)$. Thus, on substituting (B 21) and (B 22) into (B 18), one finds for the $(n = 1)$ contribution

$$\epsilon_{a,ij}(n=1) = \tfrac{1}{2}\pi(mc)^3\left(\frac{\omega_p}{\omega}\right)^2\frac{\omega_c}{\omega}\frac{1}{[N_\parallel^2+(\omega_c/\omega)^2-1]^{\frac{1}{2}}}$$
$$\times\{[H_{ij}(n=1)\,h]_{\bar{p}_\parallel=\bar{p}_\parallel^{(+)}(n=1)}+[H_{ij}(n=1)\,h]_{\bar{p}_\parallel=\bar{p}_\parallel^{(-)}(n=1)}\}, \quad \text{(B 23)}$$

with $H_{ij}(n = 1)$ defined in (B 22).

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