AS2001
Nucleosynthesis and the Chemical Evolution of the Universe
Tutorial 1 – Answers

Question 1

From equation (1) we have

$$\begin{aligned}
\rho a^3 &= \frac{3}{8\pi G} a(\dot{a}^2 + kc^2) \\
\frac{\mathrm{d}}{\mathrm{d}t}(\rho a^3) &= \frac{3}{8\pi G}\left[\dot{a}(\dot{a}^2 + kc^2) + 2a\dot{a}\ddot{a}\right]
\end{aligned}$$

and using equation (2)

$$\begin{aligned}
&= \frac{3}{8\pi G}\left[\dot{a}(-\frac{8\pi G}{c^2}a^2 p - 2a\ddot{a}) + 2a\dot{a}\ddot{a}\right] \\
&= -\frac{3}{c^2}a^2\dot{a}p.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}a}(\rho a^3) &= -\frac{3}{c^2}a^2 p \\
\frac{\mathrm{d}\rho}{\mathrm{d}a} &= -\frac{3}{a}(\rho + \frac{p}{c^2}).
\end{aligned}$$

Assuming that $p = wc^2\rho$ we have

$$\frac{\mathrm{d}\rho}{\mathrm{d}a} = -\frac{3}{a}\rho(1+w)$$

If $\rho = Ka^{-3(1+w)}$ (where K is a constant) then

$$\frac{\mathrm{d}\rho}{\mathrm{d}a} = -3(1+w)Ka^{-3(1+w)-1} = -3(1+w)\frac{\rho}{a}$$

which is just the same as in the equation above.

If $w = -1$ then

$$\rho_{\mathrm{vac}} \propto a^{-3(1-1)} = \mathrm{const},$$

i.e. ρ_{vac} remains constant during the evolution of the universe.

From equation (1) and the lecture notes we find that

$$\dot{a}^2 = \underbrace{\frac{8\pi G}{3}\rho_{\mathrm{R0}} a_0^4}_{\text{const}} \frac{1}{a^2} + \underbrace{\frac{8\pi G}{3}\rho_{\mathrm{NR0}} a_0^3}_{\text{const}} \frac{1}{a} - \underbrace{kc^2}_{\text{const}} + \underbrace{\frac{8\pi G}{3}\rho_{\mathrm{vac}}}_{\text{const}} a^2.$$

As t increases so does a and eventually the vacuum energy term will dominate over the radiation (R), non-relativistic (NR) and curvature (kc^2) terms. So for $t \to \infty$ we can neglect these terms. We then have

$$H = \frac{\dot{a}}{a} = \sqrt{\frac{8\pi G}{3}\rho_{\mathrm{vac}}} = \text{const}$$

and

$$\dot{a} = Ha.$$

The solution to this equation is

$$a(t) \propto e^{Ht}$$

and therefore a universe with $\rho_{\mathrm{vac}} \neq 0$ will eventually expand exponentially!

Subtracting equation (2) from equation (1) we have

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + \frac{3p}{c^2})$$

which is always < 0 as long as $p \geq 0$. Thus the attractive gravitational effect of matter and radiation (which have $p \geq 0$) tries to slow down the expansion of the universe ($\ddot{a} < 0$). However, because vacuum energy has $p < 0$ we get $\ddot{a} > 0$ and hence an *accelerated* expansion. Thus vacuum energy seems to have a *repulsive* gravitational effect.

Question 2

The age of the universe is approximately $t_0 \approx H_0^{-1}$. Thus the size of the observable universe is just c/H_0. If we divide equation (1) by c^2 then it has the dimension of $1/\text{length}^2$. The following quantity thus gives a characteristic length scale over which the effects of geometry should become apparent:

$$\sqrt{\frac{3c^2}{8\pi G\rho_0}} = \sqrt{\frac{3c^2}{8\pi G\rho_{c0}\Omega_0}} = \frac{c}{H_0}\frac{1}{\sqrt{\Omega_0}}$$

If $\Omega_0 \approx 1$ then this scale is approximately equal to the size of the observable universe and therefore it is not surprising that geometric effects are difficult to observe in the nearby universe.

Question 3

Radius of a black hole of mass m (Schwarzschild radius) $= 2Gm/c^2$.

$$R = \frac{2 \times 6.672 \times 10^{-11}\mathrm{Nm^2kg^{-2}} \times 1.989 \times 10^{30}\mathrm{kg}}{(2.998 \times 10^8 \mathrm{ms^{-1}})^2} \approx \frac{2 \times 7 \times 2}{9} 10^{-11+30-16}\mathrm{m} \approx 3 \text{ km}$$

Setting $\lambda = R$ (de Broglie wavelength = Schwarzschild radius) gives

$$\frac{\hbar}{m_{\rm Pl} c} = \frac{G m_{\rm Pl}}{c^2}$$

and solving for $m_{\rm Pl}$ (Planck mass) we get

$$m_{\rm Pl} = \sqrt{\frac{\hbar c}{G}} = 2.17 \times 10^{-8} \text{ kg}$$

and

$$E_{\rm Pl} = m_{\rm Pl} c^2 = \sqrt{\frac{\hbar c^5}{G}} = 1.22 \times 10^{19} \text{ Gev}$$

(1 eV = 1.602×10^{-19} J)

$$l_{\rm Pl} = R(m_{\rm Pl}) = \frac{G m_{\rm Pl}}{c^2} = \sqrt{\frac{G\hbar}{c^3}} = 1.62 \times 10^{-35} \text{ m}$$

$$t_{\rm Pl} = \frac{l_{\rm Pl}}{c} = \sqrt{\frac{G\hbar}{c^5}} = 5.39 \times 10^{-44} \text{ s}$$

Above we argued from GR to quantum gravity: the Schwarzschild radius is the length scale over which the geometric effects of a mass m are important. Starting from a 'regular' stellar mass black hole we find that it lies in the domain of GR and that we can completely neglect quantum effects. We then reduce the mass until the length scale has shrunk to the point where quantum effects are important. The 'opposite direction' argument begins with a typical elementary particle, say an electron, which lies in the domain of quantum mechanics and whose mass is so small that we can neglect its gravitational effect, i.e. the geometrical distortion of the surrounding space-time happens on a much smaller scale than the particle's de Broglie wavelength. If we increase the particle's mass then its de Broglie wavelength decreases until at some point gravitational effects are important.