AS2001
Nucleosynthesis and the Chemical Evolution of the Universe
Tutorial 2 – Answers

Question 1

$$\begin{aligned}
\epsilon_\nu \mathrm{d}\nu &= \int_0^{2\pi}\int_0^{\pi} E f(\mathbf{p})\, p^2\, \mathrm{d}p\, \sin\theta \mathrm{d}\theta\, \mathrm{d}\phi \\
&= \frac{4\pi g}{(2\pi\hbar)^3} \frac{E p^2\, \mathrm{d}p}{\exp\left(\frac{E}{kT}\right) - 1} \\
&= \frac{8\pi}{h^3} \frac{h\nu (h\nu/c)^2 h/c\, \mathrm{d}\nu}{\exp\left(\frac{h\nu}{kT}\right) - 1} \\
&= \frac{8\pi h}{c^3} \frac{\nu^3\, \mathrm{d}\nu}{\exp\left(\frac{h\nu}{kT}\right) - 1}
\end{aligned}$$

Use the substitution $x = \frac{h\nu}{kT}$.

$$\begin{aligned}
\epsilon &= \int_0^\infty \epsilon_\nu\, \mathrm{d}\nu = \frac{8\pi h}{c^3} \int_0^\infty \frac{\left(\frac{xkT}{h}\right)^3 \frac{kT}{h}\, \mathrm{d}x}{\exp(x) - 1} \\
&= \frac{8\pi}{c^3 h^3} (kT)^4 \int_0^\infty \frac{x^3\, \mathrm{d}x}{\exp(x) - 1} \\
&= \frac{8\pi^5}{15 c^3 h^3} (kT)^4 = \frac{\pi^2 k^4}{15 c^3 \hbar^3}\, T^4 = \alpha T^4,
\end{aligned}$$

where α is the radiation constant.

$$\begin{aligned}
n &= \int f(\mathbf{p}) \mathrm{d}^3 p = \frac{8\pi}{(2\pi\hbar)^3} \int_0^\infty \frac{(E/c)^2\, \mathrm{d}E/c}{\exp\left(\frac{E}{kT}\right) - 1} \\
&= \frac{1}{\pi^2 \hbar^3 c^3} (kT)^3 \int_0^\infty \frac{x^2\, \mathrm{d}x}{\exp(x) - 1} \\
&= \frac{2\zeta(3) k^3}{\pi^2 \hbar^3 c^3}\, T^3
\end{aligned}$$

$$\langle E \rangle = \frac{\epsilon}{n} = \frac{\pi^2 k^4}{15 c^3 \hbar^3}\, T^4 \frac{\pi^2 \hbar^3 c^3}{2\zeta(3) k^3}\, T^{-3} = \frac{\pi^4}{30\zeta(3)}\, kT \approx 2.7\, kT$$

Since η is conserved during the expansion we have (m_p = proton mass):

$$\eta = \frac{n_B}{n_\gamma} = \frac{n_{B0}}{n_{\gamma 0}} = \frac{\rho_{c0}\Omega_B}{m_p} \frac{\pi^2 \hbar^3 c^3}{2\zeta(3) k^3} T_0^{-3}$$

$$
\begin{aligned}
&= \left(\frac{\hbar c}{kT_0}\right)^3 \frac{3\pi H_0^2 \Omega_B}{16\zeta(3) G m_p} \\
&= 2.74 \times 10^{-8}\, \Omega_B h^2,
\end{aligned}
$$

where $\Omega_B = \frac{\rho_B 0}{\rho_{c0}}$, $\rho_{c0} = \frac{3H_0^2}{8\pi G}$ is today's value of the critical density and $H_0 = 100\, h$ km/s/Mpc.

Since there are $\approx 10^9$ photons for every baryon, there are enough high energy photons in the tail of the black-body distribution to keep deuterium from forming. The condition for deuterium to be able to form is roughly $n_\gamma(h\nu > 2.2\,\mathrm{MeV}) < n_B$. This does not occur until kT is well below 2.2 MeV.