Tides, Rotation or Anisotropy?
New Self-consistent Nonspherical Models
for Globular Clusters

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Beyond the traditional paradigm of GCs: Theoretical Motivations

Spherical King models provide a successful zeroth-order interpretation of GCs

But more realistic equilibrium models should include ...

- **External tidal field:**
  Energy truncation in King models is imposed heuristically to mimic the role of tides, but the triaxial tidal field should be imposed self-consistently

- **Internal rotation:**
  Present-day GCs are only slowly rotating. But in the past?
  Solid-body or differential?

- **Anisotropy in the velocity space:**
  Quasi-relaxed systems are expected to be approximately isotropic, but:
  - less-relaxed objects may keep memory of their formation process.
  - evolution in a (variable) tidal field may produce non-trivial kinematical signatures.

**Deviations from spherical symmetry are induced!**

Physical origin of the observed flattening? van den Bergh AJ 2008
Beyond the traditional paradigm of GCs: Observational Motivations

- New measurements of shapes and sizes of 116 GGCs are available
  Chen & Chen ApJ 2010

- Existence of the “extra-tidal light” is frequently reported
  e.g. McLaughlin & van der Marel ApJ 2005, Jordi & Grebel A&A 2010

- Proper motions of thousands of stars have been measured in selected GCs
  ω Cen e.g. Anderson & van der Marel 2010
  47 Tuc e.g. McLaughlin et al ApJS 2006

- New kinematical measurements (velocity dispersion profile, rotation curve) are available
  e.g. Sollima et al MNRAS 2009, Lane et al 2009, 2010

Renewed modeling efforts are needed
Triaxial Tidal Models
Tidal models: Definitions

- Distribution function

\[ f_K(E) = \begin{cases} 
  A \left[ \exp(-aE) - \exp(-aE_0) \right] & \text{if } E \leq E_0 \\
  0 & \text{if } E > E_0 
\end{cases} \]

\[ E = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \Phi_c \]

King AJ 1966
Tidal models: Definitions

- Distribution function

\[
f_K(H) = \begin{cases} 
    A \left[ \exp(-aH) - \exp(-aH_0) \right] & \text{if } H \leq H_0 \\
    0 & \text{if } H > H_0
\end{cases}
\]

- Tidal approximation

\[
\Phi_T(r) = \frac{1}{2} \Omega^2 (z^2 - \nu x^2) \quad \nu \equiv 4 - \frac{\kappa^2}{\Omega^2}
\]

Concentration \iff \ W_0 \equiv \psi(0) \quad \text{Tidal strength} \iff \ \epsilon \equiv \frac{\Omega^2}{4\pi G \rho_0}

- Two domains separated by the boundary surface of the configuration, defined by \( \psi(r) = 0 \), which is unknown \textit{a priori}.

\[
\hat{\nabla}^2 \psi = -9 \left[ \frac{\hat{\rho}(\psi)}{\hat{\rho}(W_0)} + \epsilon (1 - \nu) \right] \quad \text{for } \psi > 0 \quad (\text{Poisson})
\]

\[
\hat{\nabla}^2 \psi = -9\epsilon (1 - \nu) \quad \text{for } \psi < 0 \quad (\text{Laplace})
\]

Elliptical PDE in a free boundary problem
Tidal models: Perturbation method

- Tidal effect = (small) perturbation acting on the configuration described by the spherical King models: \( \epsilon \ll 1 \)

\[
\psi(\hat{r}; \epsilon) = \sum_{k=0}^{\infty} \frac{1}{k!} \psi_k(\hat{r}) \epsilon^k
\]

- Expansion of the general term of the series \( \psi_k(\hat{r}) \) in spherical harmonics \( \rightarrow \) one-dimensional (radial) Cauchy problems.

- **This perturbation problem is singular!**
  The convergence radius of the asymptotic series vanishes \( \hat{r} \rightarrow \hat{r}_{tr} \), i.e. the validity of the expansion breaks down when \( \psi_0 = O(\epsilon) \).
  - Introduction of an intermediate region (boundary layer)
  - Asymptotic matching à la Van Dyke for \( (\psi^{(\text{int})}, \psi^{(\text{lay})}) \) and \( (\psi^{(\text{lay})}, \psi^{(\text{ext})}) \)

  Van Dyke, Perturbation Methods in Fluid Mechanics, 1975

- Inspiration: rigidly rotating polytropes
  Chandrasekhar MNRAS 1933, ... Smith Ap&SS 1975

- Full explicit solution to two orders in \( \epsilon \).

- **By induction**, the \( k \)-th order solution \( \psi^{(k)} \) contains only the \( l = 0, 2, \ldots, 2k \) harmonics with even \( m \).
Two tidal sub-critical regimes

Extension parameter: $\delta_e \equiv \frac{x_e}{r_J}$

Weak deformation: $\delta_e \ll 1$
Two tidal sub-critical regimes

Extension parameter: \( \delta_e \equiv \frac{x_e}{r_J} \)

Strong deformation: \( \delta_e \approx 1 \)
Deformation shaped by the tidal potential:
- compression along $\hat{z}$
- elongation along $\hat{x}$

\[
e = [1 - (\hat{c}/\hat{a})^2]^{1/2}
\eta = [1 - (\hat{b}/\hat{a})^2]^{1/2}
\hat{a} \geq \hat{b} \geq \hat{c}
\]

$e_0, \eta_0 = \mathcal{O}(\epsilon^{1/2})$
Non-trivial!

Quadrupole moments calculated analytically!
\[
Q_{ij}^{(2)} = Q_{ij,1}\epsilon + Q_{ij,2}\frac{\epsilon^2}{2} = \int_V (3x_i x_j - r^2 \delta_{ij}) \rho(r) d^3r
\]
Axisymmetric models with:

i) solid-body rotation
ii) differential rotation
If total angular momentum is non-vanishing, in the Maxwell-Boltzmann distribution function:

\[ E \rightarrow H = E - \omega J_z \]

where \( \omega \) represents the (solid-body) angular velocity of the system.

Distribution function:

\[
f_K(H) = \begin{cases} 
A \left[ \exp (-aH) - \exp (-aH_0) \right] & \text{if } H \leq H_0 \\
0 & \text{if } H > H_0 
\end{cases}
\]

\[ H = \frac{1}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \Phi_{cen} + \Phi_C \quad \Phi_{cen}(r) = -\frac{1}{2} \omega^2 (x^2 + y^2) \]

\[ \psi(r) = a \{ H_0 - [\Phi_c(r) + \Phi_{cen}(r)] \} \]

Concentration \( \leftrightarrow \) \( W_0 \equiv \psi(0) \) \quad Rotation strength \( \leftrightarrow \) \( \chi \equiv \frac{\omega^2}{4\pi G \rho_0} \)

Formally, the same singular perturbation problem - reduced to 2D.

Deformation shaped by the centrifugal potential:
- “elongation” on ($\hat{x}, \hat{y}$)

$$e = [1 - (\hat{b}/\hat{a})^2]^{1/2}$$
$$\hat{a} \geq \hat{b}$$
$$e_0 = O(\chi^{1/2})$$
Non-trivial!

Quadrupole moments calculated analytically!
$$\hat{Q}_{zz}/\hat{Q}_{xx} = -2$$
$$\hat{Q}_{yy}/\hat{Q}_{xx} = 1$$
for every $\chi$ and $W_0$
New family in which internal rotation is rigid in the center but vanishes in the outer parts of the system, where the truncation on the energy $E$ is effective.

\[
I(E, J_z) \equiv E - \frac{\omega J_z}{1 + b J_z^2 c}
\]

\[
I \sim H = E - \omega J_z \quad \text{for low } |J_z|
\]

\[
I \sim E \quad \text{for high } |J_z|
\]

Distribution function:

\[
f_W(I) = \begin{cases} 
    A \exp(-aE_0) \left\{ \exp[-a(I - E_0)] - 1 + a(I - E_0) \right\} & \text{if } E \leq E_0 \\
    0 & \text{if } E > E_0
\end{cases}
\]

Continuous truncation in phase space

Solution of the Poisson equation with iteration method:

\[
\hat{\nabla}^2 \psi^{(n)} = -\frac{9}{\hat{\rho}_0} \hat{\rho} \left( \hat{r}, \theta, \psi^{(n-1)} \right)
\]

$\psi(\hat{r}) = a[E_0 - \Phi_c(\hat{r})]$ sets the boundary
Differentially Rotating Models: General Properties

- Dimensionless parameters:
  - Concentration: \( W_0 \equiv \psi(0) \)
  - Central solid-body rotation: \( \bar{\omega}^2 \equiv \frac{\omega^2}{4\pi G \rho_0} \)
  - Differential character: \( \bar{b} \equiv \left( \frac{9b^{1/c}a^{-2}}{4\pi G \rho_0} \right)^c \)

- For any \( W_0 \) there exists a \( \bar{\omega}_{\text{max}} \), above which models cannot be constructed because the procedure does not converge.

- Non-monotonic polar eccentricity profile

- In the center, isotropy and solid-body rotation:

- In the outer parts, transition to tangential anisotropy and no rotation

- Rapidly rotating models exhibit a toroidal core iff (NSC!)

\[
\frac{\langle \hat{v}_\phi \rangle^2}{\hat{r}^2} + \frac{1}{\hat{r}} \frac{\partial \psi}{\partial \hat{r}} > 0 \quad \hat{r} \to 0
\]

\[
\bar{\omega}^2 > \frac{1}{3} + \frac{C_2}{18} \sqrt{\frac{5}{2}}
\]

\[
C_2 = \frac{18}{5 e W_0 \gamma(5/2, W_0)} \int_0^{\hat{r}_c} d\hat{r}' \hat{\rho}_2(\hat{r}')/\hat{r}'
\]

Differentially rotating models: Examples

$W_0 = 2 \quad \tilde{\omega}/\tilde{\omega}_{max} \in [0.1, 1]$
Differentially rotating models: Examples

\[ \log(\rho/\rho_0) \]

\[ W_0 = 2 \quad \bar{\omega}/\bar{\omega}_{max} \in [0.1, 1] \]

\[ < \hat{v}_\phi > (\hat{r}, \theta) = 3\bar{\omega} \sin \theta \hat{r} + O(\hat{r}^3) \]
Differentially rotating models: Examples

Varri & Bertin in prep.

Moderate rotation

\[ \frac{\bar{\omega}}{\bar{\omega}_{max}} = 0.1 \]

Rapid rotation

\[ \frac{\bar{\omega}}{\bar{\omega}_{max}} = 0.2 \]

\[ W_0 = 2 \]

Sections in meridional plane of isodensity surfaces
Differentially rotating models: Dynamical stability

- **Starlab** Portegies Zwart et al MNRAS 2001
- \( N = 65536 \)
- Isolated models
- Single mass, no stellar evolution

☑ Models with moderate rotation are stable

\[ t = \frac{K_{ord}}{|W|} \quad W_0 = 6 \]

![Graph showing the stability of differentially rotating models](image-url)
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- Models with moderate rotation are stable
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- Extreme rotation regime is unstable
- Consistent with Ostriker & Peebles (1973) criterion! $t = 0.14 \pm 0.03$
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$$t = K_{ord}/|W|$$

$W_0 = 2$

![Graph showing the relationship between $t$ and $W$](image-url)
Differentially rotating models: Long term evolution

- Starlab
- $N = 16384$
- $W_0 = 6$
- Moderate/rapid rotation
- Isolated models
- Single mass, no stellar evolution
- No primordial binaries

Rotation accelerates the dynamical evolution

**Differentially rotating models: Long term evolution**

\[ t = \frac{K_{\text{ord}}}{|W|} \]

- **Starlab**
- \( N = 16384 \)
- \( W_0 = 6 \)
- Moderate/rapid rotation
- Isolated models
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Rotation still present in the post-core collapse phase
Conclusions

- Triaxial tidal models have been constructed as an extension of spherical King models; intrinsic and projected properties have been given. 

- Extension of spherical King models to the case of internal solid-body rotation has been performed. 

- Promising family of differentially rotating models has been proposed. 
  Varri & Bertin almost submitted

- Numerical study of dynamical stability and long term evolution in progress. 
  Varri et al in preparation

Future work

- Rotating models: N-body simulations with tidal boundary, multimass.

- Tidal models: comparison with N-body simulations of star clusters with different degree of filling of the critical Hill surface.

- Comparison with observations: interpretation of observed flattening, “extra-tidal light”, kinematics (rotation, anisotropy).