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Feynman path-integral approach to the Aharonov-Bohm effect

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The Aharonov-Bohm effect is investigated in the Feynman path-integral formulation of quantum mechanics. We consider an idealized situation with an electron moving in a magnetic-field-free region outside a solenoid whose radius to length ratio is very small. The nonvanishing vector potential term in the Lagrangian is written as an angular-velocity-dependent potential. In order to account for a singularity due to the presence of the solenoid itself, a periodic constraint is imposed on the path integral. The propagator can then be evaluated using the polar-coordinate methods of Peak and Inomata. It is found that the propagator has the general form $K(\vec{r}'',\vec{r}';\tau) = \sum_n \chi_n K_n(\vec{r}'',\vec{r}';\tau)$ where the sum is taken over all classes of homotopic paths and the χ_n are a one-dimensional representation of the homotopy group. This is the form of the propagator as conjectured by Schulman.

I. INTRODUCTION

In recent years there have been many discussions in the literature of quantum mechanics on multiply-connected spaces.[1-3] The subject is most easily discussed from the Feynman path-integral formulation where one may consider the various possible paths to belong to distinct homotopy classes, i.e., the paths cannot be continuously deformed into each other. The propagator $K(\vec{r}'',\vec{r}';\tau)$ is said to be given by a sum of partial amplitudes

$$K(\vec{r}'',\vec{r}';\tau)=\sum_n \chi_n K_n(\vec{r}'',\vec{r}';\tau)\,, \qquad (1.1)$$

where χ_n is a one-dimensional unitary representation of the fundamental group for the nth homotopy class. $K_n(\vec{r}'',\vec{r}';\tau)$ is presumed to be calculated by summing over all possible paths within the nth homotopy class. A lucid discussion of these ideas can be found in the paper by Dowker.[3]

The Aharonov–Bohm (AB)[4] effect is an example of an electron moving in the multiply-connected space SO(2). The multiply-connectedness is due to the presence of an infinitely long and infinitesimally thin solenoid containing magnetic flux Φ. The magnetic field $\vec{B}$ outside of the cylinder is of course zero (see Fig. 1). The essence of the AB effect is that even though the electron never enters the solenoid, there are observable effects (interference)[5] due to the nonvanishing of the vector potential $\vec{A}$ outside the solenoid. The amount of the interference is dependent on the confined flux. Schulman[2] has determined that the electron propagator for this problem should have the same form as Eq. (1.1) with $\chi_n=\exp(ie\Phi n/c\hbar)$. However, there does not seem to be an explicit calculation anywhere that shows to what extent the form of Eq. (1.1) can really be obtained. Furthermore, it is not too clear what the influence of the solenoid itself will be in the limit $\Phi\to 0$. In the traditional discussions[6] of the AB effect, the vanishing of the flux seems to make the system indistinguishable from one for which the underlying space is simply connected.

In this paper we discuss the AB effect by introducing the singularity associated with the solenoid as a periodic constraint in the path integral. This method was first introduced by Edwards[7] and later refined by Inomata and Singh[8] in connection with the problem of entangled polymers. We briefly review this method in the next section. In Sec. III we perform an explicit calculation for the propagator associated with the AB effect and determine it up to a Fourier transform. In Sec. IV, in order to get some sort of closed-form result, we impose a radial constraint on the electron so as to confine it to the surface of a cylinder of radius R centered on the origin. With this extra condition the Fourier transform can be calculated in closed form. We also consider a WKB approximation on the exact result of Sec. III, which again allows us to evaluate the Fourier integral. These two results are noted to be very similar.

II. PATH INTEGRALS WITH A PERIODIC CONSTRAINT[8]

In the Feynman path-integral formulation of quantum mechanics it is asserted that the propagator is given by

$$K(\vec{r}'',\vec{r}';\tau)=\int \exp\left[\frac{i}{\hbar}S(\vec{r}'',\vec{r}')\right]D\vec{r}(t)\,, \qquad (2.1)$$

where the symbol $Dr(t)$ means the integrations are to be taken over all possible paths from $\vec{r}'=\vec{r}(0)$ to $\vec{r}''=\vec{r}(\tau)$. $S(\vec{r}'',\vec{r}')$ is, of course, the classical action

$$S(\vec{r}'',\vec{r}')=\int_0^\tau L(\vec{r},\dot{\vec{r}})\,dt\,.$$

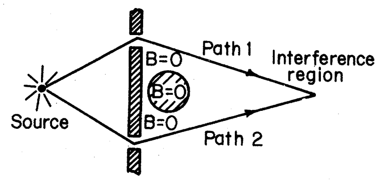

FIG. 1. An idealized experimental arrangement to illustrate the Aharonov-Bohm effect.

In this paper we will confine ourselves to path integrations over a two-dimensional plane where $\vec{r}=(r,\theta)$.

Now if there should exist a singular point in the plane (which we shall take to be at the origin) the space will become multiply-connected. A path encircling the singular point n number of times is homotopically different from one encircling it m number of times ($m\neq n$). The various nonhomotopic configurations can be classified by the number of turns about the singularity. This is the winding number n where $n=0,\pm1,\pm2,\ldots$ (n turns counterclockwise if $n\geq 0$ and $n+1$ clockwise if $n\leq -1$). (See Fig. 2.) If we let Θ ($0\leq\Theta<2\pi$) be the angle $\cos^{-1}(\vec{r}''\cdot\vec{r}')$ then we can write

$$\int_0^\tau \dot{\theta}\,dt=\Theta+2\pi n\,. \qquad (2.2)$$

To incorporate the constraint

$$\int_0^\tau \dot{\theta}\,dt=\phi \qquad (2.3)$$

into the propagator, we write

$$K_\phi(\vec{r}'',\vec{r}';\tau)=\int\delta\left(\phi-\int_0^\tau\dot{\theta}\,dt\right)\times\exp\left[\frac{i}{\hbar}S(\vec{r}'',\vec{r}')\right]D\vec{r}(t)\,. \qquad (2.4)$$

The δ function has the effect of selecting out the different classes of homotopic configurations. The total propagator will be given by

$$K(\vec{r}'',\vec{r}';\tau)=\int_{-\infty}^{\infty}K_\phi\,d\phi\,. \qquad (2.5)$$

Using the relation

$$2\pi\delta(x)=\int_{-\infty}^{+\infty}\exp(i\lambda x)\,d\lambda\,,$$

we write K_ϕ as

$$K_\phi(\vec{r}'',\vec{r}';\tau)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}K_\lambda(\vec{r}'',\vec{r}';\tau)\exp(i\lambda\phi)\,d\lambda\,, \qquad (2.6)$$

where

$$K_\lambda(\vec{r}'',\vec{r}';\tau)=\int\exp\left\{\frac{i}{\hbar}\int_0^\tau[L(\dot{\vec{r}},\vec{r})-\lambda\hbar\dot{\theta}]\,dt\right\}D\vec{r}(t)\,. \qquad (2.7)$$

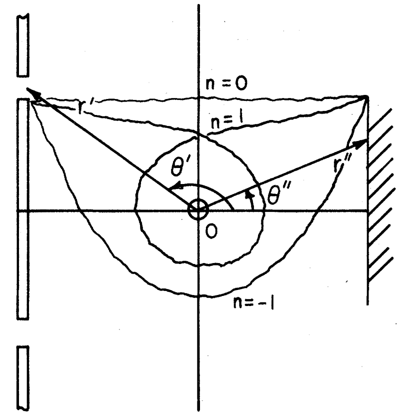

FIG. 2. The origin of our set of polar coordinates is centered on the infinitely thin solenoid. Paths from various homotopy classes are shown.

We are thus led to consider the path-integral equation (2.7) in which the original Lagrangian L has been replaced by an effect Lagrangian L' containing an angular-velocity-dependent potential,

$$L'=L-\lambda\hbar\dot{\theta}\,.$$

III. THE AHARONOV-BOHM EFFECT

We will proceed to calculate the propagator for an electron moving in the magnetic-field-free region outside of a solenoid. The Lagrangian for this system is

$$L=\tfrac{1}{2}\mu\dot{\vec{r}}^2+\frac{e}{c}\vec{A}\cdot\dot{\vec{r}}\,. \qquad (3.1)$$

For the vector potential we take

$$\vec{A}=\frac{\Phi}{2\pi}(-y\hat{\imath}+x\hat{\jmath})/(x^2+y^2)\,,$$

where Φ is the confined flux. It is easily checked that $\vec{B}=\vec{\nabla}\times\vec{A}=0$. This potential has been widely used to discuss the AB effect.[4] Following Edwards[7] we rewrite the second term in (3.1) as

$$\frac{e}{c}\vec{A}\cdot\dot{\vec{r}}=\frac{e}{c}\frac{\Phi}{2\pi}\dot{\theta}\,. \qquad (3.2)$$

Thus, we will evaluate the path integral (2.7) with the effective Lagrangian

$$L'=\tfrac{1}{2}\mu\dot{\vec{r}}^2+(\alpha-\lambda\hbar)\dot{\theta}\,,$$

where we have set $e\Phi/2\pi c=\alpha$.

Now, using the customary definition of the path integral, we write Eq. (2.7) as

$$K_\lambda(\vec{r}'',\vec{r}';\tau)=\lim_{N\to\infty}A_N\int\exp\left[\frac{i}{\hbar}\sum_{j=1}^{N}S'(\vec{r}_j,\vec{r}_{j-1})\right]\prod_{j=1}^{N-1}\times(d\vec{r}_j)\,, \qquad (3.3)$$

where $\vec{r}_j = \vec{r}(t_j)$, $\vec{r}_0 = \vec{r}'$, $\vec{r}_N = \vec{r}''$, and $t_j - t_{j-1} = \tau/N = \epsilon$. The partial action for a small time interval ϵ may be expressed by

$$S'(\vec{r}_j, \vec{r}_{j-1}) \simeq \epsilon L'(\Delta\vec{r}_j/\epsilon, \vec{r}_j) .$$

In polar coordinates we can write this as[8,10]

$$S'(\vec{r}_j, \vec{r}_{j-1}) = \frac{\mu}{2\epsilon}(r_j^2 + r_{j-1}^2) + \frac{\mu}{\epsilon} r_j r_{j-1} \cos(\theta_j - \theta_{j-1}) + (\alpha - \lambda\hbar)(\theta_j - \theta_{j-1}) . \tag{3.4}$$

The exponential of Eq. (3.4) needed in Eq. (3.3) is

$$\exp\left[\frac{i}{\hbar} S'(\vec{r}_j, \vec{r}_{j-1})\right] = \exp\left\{\frac{i\mu}{2\epsilon\hbar}(r_j^2 + r_{j-1}^2) - \frac{i\mu}{\hbar\epsilon} r_j r_{j-1}\left[\cos(\theta_j - \theta_{j-1}) + \frac{\hbar\epsilon}{\mu r_j r_{j-1}} \beta(\theta_j - \theta_{j-1})\right]\right\}, \tag{3.5}$$

where we have set $\beta = \lambda - \alpha/\hbar$ for now. In order to take into account contributions up to fourth order in $\Delta\theta$, we employ the expansions

$$\cos(\Delta\theta) - a\epsilon\Delta\theta \simeq \cos(\Delta\theta + a\epsilon) + \tfrac{1}{2} a^2\epsilon^2 \tag{3.6}$$

and

$$\exp\left[\frac{u}{\epsilon} \cos\theta\right] = \sum_{m=-\infty}^{+\infty} \exp(im\theta) I_m\left(\frac{u}{\epsilon}\right), \tag{3.7}$$

for small ϵ and for $|\arg(u/\epsilon)| < \pi/2$.[11] The exponential (3.5) now becomes

$$\exp\left[\frac{i}{\hbar} S'(\vec{r}_j, \vec{r}_{j-1})\right] = \sum_{m_j=-\infty}^{+\infty} R_{(m_j+\beta)}(r_j, r_{j-1}) \exp[im_j(\theta_j - \theta_{j-1})], \tag{3.8}$$

where we have also made use of the asymptotic expression for the modified Bessel function

$$I_m\left(\frac{u}{\epsilon}\right) \simeq \left(\frac{\epsilon}{2\pi u}\right)^{1/2} \exp\left[\frac{u}{\epsilon} - \tfrac{1}{2}(m^2 - \tfrac{1}{4})\frac{\epsilon}{u} + O(\epsilon^2)\right] \tag{3.9}$$

and defined

$$R_{m_j+\beta}(r_j, r_{j-1}) = \exp\left[\frac{i\mu}{2\epsilon\hbar}(r_j^2 + r_{j-1}^2)\right] I_{(m_j+\beta)}\left(\frac{-i\mu r_j r_{j-1}}{\epsilon\hbar}\right).$$

The propagator K_λ then becomes

$$K_\lambda(r'', \theta'', r', \theta'; \tau) = \lim_{N\to\infty} A_N \int \cdots \int \prod_{j=1}^{N} \sum_{m_j=-\infty}^{+\infty} R_{(m_j+\beta)}(r_j, r_{j-1}) \exp[im_j(\theta_j - \theta_{j-1})] \prod_{j=1}^{N-1} (r_j\, dr_j\, d\theta_j), \tag{3.10}$$

where $r_0 = r'$, $r_N = r''$, $\theta_0 = \theta'$, and $\theta_N = \theta''$. After interchanging the multiplication and summations, the angular integrations can be performed with the help of the orthogonality relation

$$\frac{1}{2\pi}\int_0^{2\pi} \exp[i(m' - m)\theta]\, d\theta = \delta_{mm'} .$$

We obtain

$$K_\lambda(r'', \theta'', r', \theta'; \tau) = \sum_{m=-\infty}^{+\infty} \exp[im(\theta'' - \theta')] Q_{(m+\beta)}(r'', r'; \tau), \tag{3.11}$$

where

$$Q_{(m+\beta)}(r'', r'; \tau) = \lim_{N\to\infty} (2\pi)^{N-1} A_N \int \cdots \int \prod_{j=1}^{N} R_{(m_j+\beta)}(r_j, r_{j-1}) \prod_{j=1}^{N-1} (r_j\, dr_j) . \tag{3.12}$$

Now resubstituting $\beta = \lambda - \alpha/\hbar$, and using the definition (2.6), we find

$$K_\phi(\vec{r}'', \vec{r}'; \tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \exp[im(\theta'' - \theta') + i\lambda\phi] Q_{(m+\lambda-\alpha/\hbar)}(r'', r'; \tau)\, d\lambda .$$

Making the change of variables $\lambda \to \lambda - m + \alpha/\hbar$, we have

$$K_\phi(\vec{r}'', \vec{r}'; \tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \exp\left[im(\theta'' - \theta' - \phi) + i\left(\lambda + \frac{\alpha}{\hbar}\right)\phi\right] Q_\lambda(r'', r'; \tau)\, d\lambda .$$

Using the identity

$$\sum_{m=-\infty}^{+\infty} \exp(im\theta) = 2\pi \sum_{n=-\infty}^{+\infty} \delta(\theta + 2\pi n),$$

we get

$$K_\phi(\vec{r}'', \vec{r}'; \tau) = \sum_{n=-\infty}^{+\infty} \delta(\theta'' - \theta' - \phi + 2\pi n) \int_{-\infty}^{+\infty} \exp\left[i\left(\lambda + \frac{\alpha}{\hbar}\right)\phi\right] Q_\lambda(r'', r'; \tau)\, d\lambda. \qquad (3.13)$$

The δ function is selecting out homotopic paths if we interpret n as the winding number. The total propagator is

$$K(\vec{r}'', \vec{r}'; \tau) = \int_{-\infty}^{+\infty} K_\phi\, d\phi = \sum_{n=-\infty}^{+\infty} K_n(\vec{r}'', \vec{r}'; \tau), \qquad (3.14)$$

where

$$K_n(\vec{r}'', \vec{r}'; \tau) = \exp\left[\frac{i\alpha}{\hbar}(\theta'' - \theta' + 2\pi n)\right] \times \int_{-\infty}^{+\infty} \exp[i\lambda(\theta'' - \theta' + 2\pi n)] Q_\lambda\, d\lambda. \qquad (3.15)$$

We can express Eq. (3.12) in the form

$$K(\vec{r}'', \vec{r}'; \tau) = \exp\left[\frac{i}{\hbar}\alpha(\theta'' - \theta')\right] \times \sum_{n=-\infty}^{+\infty} \exp\left[\frac{i\alpha}{\hbar} 2\pi n\right] \tilde{K}_n, \qquad (3.16)$$

where

$$\tilde{K}_n = \int_{-\infty}^{+\infty} \exp[i\lambda(\theta'' - \theta' + 2\pi n)] Q_\lambda\, d\lambda. \qquad (3.17)$$

The overall angle-dependent phase factor in (3.15) can be removed if we use

$$\psi(\vec{r}'', \tau) = \int K(\vec{r}'', \vec{r}'; \tau)\, \psi(\vec{r}', 0)\, d\vec{r}'$$

and define the new wave functions and propagator as

$$\psi_{\text{new}}(\vec{r}, \tau) = \exp\left(\frac{i}{\hbar}\alpha\theta\right)\psi_{\text{old}}(\vec{r}, \tau),$$

$$K_{\text{new}}(\vec{r}'', \vec{r}'; \tau) = \exp\left[-\frac{i\alpha}{\hbar}(\theta'' - \theta')\right] K_{\text{old}}(\vec{r}'', \vec{r}'; \tau).$$

We can then write our propagator as

$$K(\vec{r}'', \vec{r}'; \tau) = \sum_{n=-\infty}^{+\infty} \exp\left[\frac{nie\Phi}{c\hbar}\right] \tilde{K}_n. \qquad (3.18)$$

This is the form as conjectured by Schulman.[2]

We have not yet evaluated the radial path integral (3.12). As this has already been done elsewhere, we only present the result which is[8]

$$Q_\lambda(r'', r'; \tau) = \left(\frac{\mu}{2\pi i\hbar\tau}\right) \exp\left[\frac{i\mu}{2\hbar\tau}(r'^2 + r'')\right] \times I_\lambda\left(\frac{-i\mu r' r''}{\hbar\tau}\right),$$

where the normalization constant $A_N = (\mu/2\pi i\epsilon\hbar)^N$. Equation (3.15) now becomes

$$\tilde{K}_n(\vec{r}'', \vec{r}'; \tau) = \left(\frac{\mu}{2\pi i\hbar\tau}\right) \exp\left[\frac{i\mu}{2\hbar\tau}(r'^2 + r'')\right] \times \int_{-\infty}^{+\infty} \exp[i\lambda(\theta'' - \theta' + 2\pi n)] I_\lambda \times \left(\frac{-i\mu r' r''}{\hbar\tau}\right) d\lambda. \qquad (3.19)$$

IV. EVALUATION OF $\tilde{K}_n$

We have obtained the partial amplitude $\tilde{K}_n$ as a Fourier transform in Eq. (3.19). Owing to the complicated dependence on λ of the modified Bessel function, we cannot obtain a closed form. To get some idea of the behavior of $\tilde{K}_n$ we will first consider the special case of the electron with a radial constraint, i.e., constrained to the surface of a cylinder of radius R centered on the origin.

In order to impose the radial constraint, we go back to Eq. (3.12) and insert $\delta(\vec{r}_j - \vec{R})$ in all the $N-1$ integrals. This will yield[8]

$$Q_\lambda = \frac{1}{(4\pi^2 I)^{1/2}} \exp(-i\lambda\hbar\tau/2I), \qquad (4.1)$$

where $I = R^2\mu$ and the normalization constant has been chosen as

$$A_N = R\left[\frac{2\pi i\hbar\epsilon}{\mu} \exp(-i\hbar\epsilon/4I)\right]^{-N/2}.$$

Now using Eqs. (4.1) and (3.16) we get

$$\tilde{K}_n = \left(\frac{2\pi i\hbar\tau}{I}\right)^{-1/2} \exp[iI(\theta'' - \theta' + 2\pi n)^2/2\hbar\tau]. \qquad (4.2)$$

The total propagator will now be

$$K(\theta'', \theta'; \tau) = \left(\frac{2\pi i\hbar\tau}{I}\right)^{-1/2} \times \sum_{n=-\infty}^{+\infty} \exp\left(\frac{ie\Phi n}{c\hbar}\right) \times \exp\left[\frac{iI}{2\hbar\tau}(\theta'' - \theta' + 2\pi n)^2\right]. \qquad (4.3)$$

This is the same form of the propagator for a rigid rotator as obtained by Schulman,[12] except that there is an additional phase factor dependent on the magnetic flux. Now, to examine the propagator without the imposition of extra constraints, we evaluate Eq. (3.18) using a WKB approximation. We use the asymptotic expansion (3.9) but now consider the expression valid for $\hbar \to 0$ instead of ϵ very small. In this approximation Eq. (3.18) becomes

$$\tilde{K}_n(\vec{r}'', \vec{r}'; \tau) = \left(\frac{\mu}{2\pi i\hbar\tau}\right)\left(\frac{i\hbar\tau}{2\mu r' r''}\right)^{1/2} \exp\left[\frac{i\mu}{2\hbar\tau}(r'^2 + r'') - \frac{i\hbar\tau}{\mu r' r''}\right] \int_{-\infty}^{+\infty} \exp[i\lambda(\theta'' - \theta' + 2\pi n)] \exp\left(\frac{i\hbar\tau\lambda^2}{\mu r' r''}\right) d\lambda \,. \quad (4.4)$$

Evaluating the Fresnel integral and then dropping terms in the exponential that are quadratic and higher in powers of $\hbar$ we obtain

$$\tilde{K}_n(\vec{r}'', \vec{r}'; \tau) = \left(\frac{\mu}{2\pi i\hbar\tau}\right) \exp\left[\frac{i\mu}{2\hbar\tau}(r'^2 + r''^2)\right] \times \exp\left[\frac{-i\mu r' r''}{2\hbar\tau}(\theta'' - \theta' + 2\pi n)^2\right]. \quad (4.5)$$

Thus we see that this propagator has the same angular dependence as that of the rigid rotator in Eq. (4.2). It is interesting to note that without the use of the periodic constraint as described in Sec. II, the form of the propagator in Eq. (3.18) cannot be obtained. This is because there would be no way to distinguish the propagators found by path integration over each class of homotopic paths. In fact, they would all have the form of the propagator for a free particle in two dimensions.

V. DISCUSSION

The class of problems treated exactly by the Feynman path-integral approach is very limited. Besides, the extension to non-Cartesian coordinates is not straightforward.[8,10,13] Our solution of the AB effect in polar coordinates extends the range of problems solvable directly by functional integration. A unique feature of this approach is that one can take into account the multiply-connectedness of the space induced by the solenoid itself. This shows that even if the magnetic flux confined to the solenoid should vanish, there should still be some observable effect. This could be interpreted as the scattering of the electrons from an infinitely long cylinder. As was pointed out, in the usual discussions of the AB effect, taking $\phi = 0$ gives results indistinguishable from those of a simple connected space. From topological arguments, the form of the propagator was conjectured to be that of Eq. (1.1).[2] Our calculations for obtaining Eq. (3.18) seem to bear out these arguments. It should be noted, however, that without the imposition of the periodic constraint described in Sec. II,[8] the appropriate form of the propagator cannot be obtained. One would find that there would be no way to distinguish between the propagators of different homotopy classes; they would all have the form of a free-particle propagator.

Our resulting Eq. (3.18) was given in integral form. Imposing a radial constraint on the electron allowed us to evaluate the integral and led to a result similar to that of the rigid rotator.

Finally, we feel that the approach used in this paper could be extended to problems where similarly topological features are involved. These might include the problem of electron propagation in a periodic lattice and the recent interest in topological solitons and instantons.

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